THE FINITE DIFFERENCE METHOD FOR THE PERIODIC BOUNDARY AND INITIAL VALUE PROBLEM OF A CLASS OF SYSTEM OF GENERALIZED ZAKHAROV EQUATIONS*1)

LIANG JUN-DU (梁君度)
(Zhongshan University, Guangzhou, China)

Abstract

This paper is intended to study the finite difference method for the periodic boundary and initial value problem of a class of system of generalized Zakharov equations.

1. Introduction

We consider the system of generalized Zakharov equations

$$i\varepsilon_t+\varepsilon_{xx}-\alpha(x)\eta\varepsilon+\beta(x)|\varepsilon|^2\varepsilon=0, \tag{1.1}$$

$$\eta_{tt}-\eta_{xx}+\gamma\eta_t=|\varepsilon|^2_{xx}, \tag{1.2}$$

with the periodic boundary conditions

$$\varepsilon(x, t)=\varepsilon(x+D, t),\ \eta(x, t)=\eta(x+D, t),\quad \forall x,\ t\geqslant 0, \tag{1.3}$$

and the initial conditions

$$\varepsilon(x, 0)=\varepsilon_0(x),\ \eta(x, 0)=\eta_0(x),\ \eta_t(x, 0)=\eta_1(x),\quad 0\leqslant x\leqslant D, \tag{1.4}$$

where $i=\sqrt{-1}$, $D>0$, γ is a real constant, $\alpha(x)$ and $\beta(x)$ are known functions which possess the period D, $\eta(x, t)$ is an unknown real function, $\varepsilon(x, t)=(\varepsilon_1(x, t),\cdots, \varepsilon_M(x, t))^T$ is an M-dimensional vector of complex function. When $M=1$, $\alpha(x)=1$, $\beta(x)=0$, and $\gamma=0$, the equation is the model equation presented by Zakharov[1,2], Morales and Lee[3,4] solved this equation by a finite difference method and obtained many results. Guo Bo-ling [5,6] proved the existence and uniqueness of the classical solution, and the stability of the difference scheme corresponding to the model equation. In this paper the existence, uniqueness of the difference solution and the convergence, stability of the scheme are proved theoretically.

2. Symbol and Convention. Difference Scheme

Let Q denote the rectangular region $[0, D]\times[0, T]$. Let Q_{hk} be a rectangular lattice covering Q, determined by the intersection of the coordinate lines

$$x=jh,\quad j=0, 1, \cdots, J,$$

* Received April 10, 1984.
1) I express my appreciation to Professors Guo Bo-ling and Huang You-qian for their advice and encouragement.

$$t=nk, \quad n=0,\ 1,\ \cdots,\ N,$$

where $h=D/J$ and $k=T/N$. Let x_j denote jh and t_n denote nk. We shall be concerned with functions defined on the lattice region Q_{hk}. For their forward and backward difference quotients we shall employ the following notations

$$\phi_x(x,\ t)=\frac{1}{h}[\phi(x+h,\ t)-\phi(x,\ t)]=\frac{1}{h}\Delta_+\phi,$$

$$\phi_{\bar{x}}(x,\ t)=\frac{1}{h}[\phi(x,\ t)-\phi(x-h,\ t)]=\frac{1}{h}\Delta_-\phi,$$

$$\phi_t(x,\ t)=\frac{1}{k}[\phi(x,\ t+k)-\phi(x,\ t)],$$

$$\phi_{\bar{t}}(x,\ t)=\frac{1}{k}[\phi(x,\ t)-\phi(x,\ t-k)].$$

We also introduce inner product and norms appropriate to functions defined on the lattice Q_{hk}: $(f,\ g)_h=\sum_{j=1}^{J}f(x_j)\overline{g(x_j)}h$, $\|f\|_h^2=(f,\ f)_h$, $\|f\|_{l,h}^2=\|f\|_h^2+\sum_{1\le s\le l}\left\|\frac{1}{h^s}\Delta_+^s f\right\|_h^2$, $\|f\|_{L_\infty}=\sup_{0\le j\le J}|f(x_j)|$, $\left\|\frac{1}{h^l}\Delta_+^l f\right\|_{L_\infty}=\sup_{0\le j\le J}\left|\frac{1}{h^l}\Delta_+^l f(x_j)\right|$. The norm corresponding to the space of square integrable functions is

$$\|f\|_{L_2}^2=\int_0^D|f(x)|^2dx,$$

where f is a vector valued function. Corresponding to (1.1)—(1.4), we establish the following difference scheme, denoted by $(2)^h$ or $(2)^{h'}$,

$(2)^h$

$$is_{\bar{t}}^h+s_{x\bar{x}}^h-\alpha(x)\eta^h s^h+\beta(x)|s^h|^2s^h=0, \tag{2.1}$$

$$\eta_{t\bar{t}}^h-\eta_{x\bar{x}}^h+\gamma\eta_{\bar{t}}^h=|s^h|_{x\bar{x}}^2, \tag{2.2}$$

$$s^h(x_j,\ t)=s^h(x_{j+J},\ t),\ \eta^h(x_j,\ t)=\eta^h(x_{j+J},\ t),\quad \forall j,\ t\ge0, \tag{2.3}$$

$$s^h(x_j,\ 0)=s_0(x_j),\ \eta^h(x_j,\ 0)=\eta_0(x_j),\ \eta_t^h(x_j,\ 0)=\eta_1(x_j) \tag{2.4}$$

or

$(2)^{h'}$

$$is_j^{n+1}=is_j^n-\frac{k}{h}\Delta_+\Delta_-s_j^{n+1}+k\alpha_j\eta_j^{n+1}s_j^{n+1}-k\beta_j|s_j^{n+1}|^2s_j^{n+1}, \tag{2.5}$$

$$\eta_j^{n+1}=2\eta_j^n-\eta_j^{n-1}+\frac{k^2}{h^2}\Delta_+\Delta_-\eta_j^{n+1}-k\gamma(\eta_j^{n+1}-\eta_j^n)+\frac{k^2}{h^2}\Delta_+\Delta_-|s_j^{n+1}|^2, \tag{2.6}$$

$$s_j^n=s_{j+J}^n,\ \eta_j^n=\eta_{j+J}^n,\quad \forall j,\ n\ge0, \tag{2.7}$$

$$s_j^0=s_0(x_j),\ \eta_j^0=\eta_0(x_j),\ \eta_j^0-\eta_j^{-1}=k\eta_1(x_j). \tag{2.8}$$

3. Existence and Uniqueness of Difference Solution

We can regard the problem $(2)^{h'}$ as a nonlinear system of unknowns s_j^{n+1} and $\eta_j^{n+1}(j=1,\ \cdots,\ J)$, but s_j^n and η_j^n, $\eta_j^{n-1}(j=1,\ \cdots,\ J)$ are known. We shall prove the existence and uniqueness of the solution for the system $(2)^{h'}$.

Lemma 1. *For two arbitrary vectors $\{u_j\}$ and $\{v_j\}$ $(j=1,\ \cdots,\ J)$, the identity*

$$\sum_{j=1}^{J}u_j\Delta_+\Delta_-v_j=-\sum_{j=1}^{J}\Delta_+u_j\Delta_+v_j-u_1(v_1-v_0)+u_{J+1}(v_{J+1}-v_J)$$

holds.

Theorem 1. *Suppose that one of the following conditions are satisfied*

(1) $\gamma>0$,

(2) $\gamma<0,\ 1+2\gamma K>0$.

Then there exists a solution ε_j^{n+1}, η_j^{n+1} $(j=1, \cdots, J;\ n=1, \cdots, N)$ *of the difference system* $(2)^{h'}$.

Proof. Let E_1 be an $M\times J$-dimensional complex Euclidean space, and E_2 a J-dimensional real Euclidean space, $E=E_1\times E_2$. For the $M+1$-dimensional vector $X_j=\begin{pmatrix} Z_j \\ Y_j\end{pmatrix}_1^M (j=1, \cdots, J)$, we construct an $M+1$-dimensional vector $W_j=\begin{pmatrix} E_j \\ N_j\end{pmatrix}_1^M$ $(j=1, \cdots, J)$,

$$iE_j=i\varepsilon_j^n-\lambda\frac{k}{h^2}\Delta_+\Delta_-Z_j-\lambda k\alpha_jY_jZ_j-\lambda k\beta_j|Z_j|^2Z_j, \tag{3.1}$$

$$j=1, \cdots, J,$$

$$N_j=2\eta_j^n-\eta_j^{n-1}+\lambda\frac{k^2}{h^2}\Delta_+\Delta_-Y_j-\lambda\gamma k(Y_j-\eta_j^n)+\lambda\frac{k^2}{h^2}\Delta_+\Delta_-|Z_j|^2, \tag{3.2}$$

where λ is a real parameter. We set $X_j=X_{j+J}$, so that $W_j=W_{j+J}$. Therefore we have defined operators T_λ: $E\to E$. It can be easily verified that

(1) For any $\lambda\in[0, 1]$, the operators T_λ are completely continuous.

(2) For any bounded set of E, T_λ are uniformly continuous for $0\leqslant\lambda\leqslant1$.

(3) If $\lambda=0$, $T_0(E)$ is a definite element of E.

By the Leary Schauder's theorem[7], in order to prove the existence of the solution of the problem $(2)^{h'}$, it is enough to prove that any possible solution of equation

$$X=T_\lambda(X) \tag{3.3}$$

is uniformly bounded for any $0\leqslant\lambda\leqslant1$.

For this purpose, let $W_j=\begin{pmatrix} E_j \\ N_j\end{pmatrix}_1^M (j=1, \cdots, J)$ be a solution of equation (3.3), i.e.

$$iE_j=i\varepsilon_j^n-\lambda\frac{k}{h^2}\Delta_+\Delta_-E_j+\lambda k\alpha_jN_jE_j-\lambda k\beta_j|E_j|^2E_j, \tag{3.4}$$

$$N_j=2\eta_j^n-\eta_j^{n-1}+\lambda\frac{k^2}{h^2}\Delta_+\Delta_-N_j-\lambda\gamma k(N_j-\eta_j^n)+\lambda\frac{k^2}{h^2}\Delta_+\Delta_-|E_j|^2,$$

$$j=1, \cdots, J. \tag{3.5}$$

Multiplying (3.4) by $\overline{E}_j$ and summing for $j=1, \cdots, J$, we obtain

$$i\sum_{j=1}^J|E_j|^2=i\sum_{j=1}^J\varepsilon_j^n\cdot\overline{E}_j-\lambda\frac{k}{h^2}\sum_{j=1}^J(\Delta_+\Delta_-E_j)\cdot\overline{E}_j$$
$$+\lambda\frac{k}{h^2}\sum_{j=1}^J\alpha_jN_j|E_j|^2-\lambda k\sum_{j=1}^J\beta_j|E_j|^4.$$

Applying Lemma 1 and $E_j=E_{j+J}$, we have $\sum_{j=1}^J(\Delta_+\Delta_-E_j)\cdot\overline{E}_j=-\sum_{j=1}^J|\Delta_+E_j|^2$. Since α_j, β_j, N_j are real, when we take the imaginary part, there is

$$\sum_{j=1}^J|E_j|^2=\sum_{j=1}^J\mathrm{Re}(\varepsilon_j^n\cdot\overline{E}_j)\leqslant\frac{1}{2}\sum_{j=1}^J|\varepsilon_j^n|^2+\frac{1}{2}\sum_{j=1}^J|E_j|^2;$$

or

$$\sum_{j=1}^J|E_j|^2\leqslant\sum_{j=1}^J|\varepsilon_j^n|^2=M^2, \quad M=\text{const.}$$

Equation (3.5) can be written in the form

$$N_j-2\eta_j^n+\eta_j^{n-1}=\lambda\frac{k^2}{h^2}\Delta_+\Delta_-N_j-\lambda\gamma k(N_j-\eta_j^n)+\lambda\frac{k^2}{h^2}\Delta_+\Delta_-|E_j|^2.$$

Multiplying the above expression by $N_j-\eta_j^n$ and summing up for $j=1, \cdots, J$, we have

$$\sum_{j=1}^{J}(N_j-2\eta_j^n+\eta_j^{n-1})(N_j-\eta_j^n)=\lambda\frac{k^2}{h^2}\sum_{j=1}^{J}\Delta_+\Delta_-N_j(N_j-\eta_j^n)$$

$$-\lambda\gamma k\sum_{j=1}^{J}(N_j-\eta_j^n)^2+\lambda\frac{k^2}{h^2}\sum_{j=1}^{J}\Delta_+\Delta_-|E_j|^2(N_j-\eta_j^n)$$

$$\leqslant-\lambda\frac{k^2}{h^2}\sum_{j=1}^{J}|\Delta_+N_j|^2+\lambda\frac{k^2}{h^2}\sum_{j=1}^{J}(\Delta_+N_j)(\Delta_+\eta_j^n)$$

$$-\lambda\gamma k\sum_{j=1}^{J}(N_j-\eta_j^n)^2+\lambda\frac{k^2}{h^2}\sum_{j=1}^{J}\Delta_+\Delta_-|E_j|^2(N_j-\eta_j^n)$$

$$\leqslant\frac{\lambda k^2}{4h^2}\sum_{j=1}^{J}|\Delta_+\eta_j^n|^2-\lambda\gamma k\sum_{j=1}^{J}(N_j-\eta_j^n)^2$$

$$+\frac{k^2}{2\delta h^2}\sum_{j=1}^{J}(\Delta_+\Delta_-|E_j|^2)^2+\frac{k^2\delta}{2h^2}\sum_{j=1}^{J}(N_j-\eta_j^n)^2,$$

where δ is an arbitrary positive constant. From $E_j=E_{j+J}$, we can see that

$$\sum_{j=1}^{J}(\Delta_+\Delta_-|E_j|^2)^2\leqslant\sum_{j=1}^{J}(|E_{j+1}|^2+2|E_j|^2+|E_{j-1}|^2)^2\leqslant\left(4\sum_{j=1}^{J}|E_j|^2\right)^2\leqslant 16M^4.$$

It can be easily verified that

$$\sum_{j=1}^{J}(N_j-2\eta_j^n+\eta_j^{n-1})(N_j-\eta_j^n)=\frac{1}{2}\sum_{j=1}^{J}[(N_j-\eta_j^n)^2-(\eta_j^n-\eta_j^{n-1})^2]$$

$$+\frac{1}{2}\sum_{j=1}^{J}(N_j-2\eta_j^n+\eta_j^{n-1})^2.$$

Thus we obtain

$$\left(1+2\lambda\gamma k-\frac{k^2\delta}{h^2}\right)\sum_{j=1}^{J}(N_j-\eta_j^n)^2\leqslant\frac{k^2}{2h^2}\sum_{j=1}^{J}|\Delta_+\eta_j^n|^2+\sum_{j=1}^{J}(\eta_j^n-\eta_j^{n-1})^2+\frac{16k^2}{\delta h^2}M^4.$$

If $\gamma>0$, we choose $\delta<\frac{h^2}{k^2}$; then $1-\frac{k^2\delta}{h^2}>0$. If $\gamma<0$, when $1+2\gamma k>0$, we choose $\delta<\frac{h^2+2\gamma h^2k}{k^2}$; then $1+2\gamma k-\frac{k^2\delta}{h^2}>0$, $\sum_{j=1}^{J}(N_j-\eta_j^n)^2$ have uniform bounds for $0\leqslant\lambda\leqslant1$. Therefore there exists a constant K independent of λ, such that

$$|W|^2=\sum_{j=1}^{J}|E_j|^2+\sum_{j=1}^{J}|N_j|^2\leqslant K^2. \tag{3.6}$$

This completes the proof.

Theorem 2. *Assume that the following conditions and conditions of Theorem 1 are satisfied*

(1) $\max\limits_{0\leqslant x\leqslant D}|\alpha(x)|\leqslant\alpha^*$, $\max\limits_{0\leqslant x\leqslant D}|\beta(x)|\leqslant\beta^*$,

(2) $\frac{k}{h}\leqslant C$, *C is an arbitrarily given positive constant.*

When $1-k\left(|\gamma|+\frac{1}{2}\alpha^*K+2\beta^*K^2+2\alpha^*K^3C^2\right)>0$, the solution of $(2)^{h'}$ is unique.

Proof. From Theorem 1 the solution of $(2)^{h'}$ exists and is bounded. Let $W_j=\begin{pmatrix}E_j\\N_j\end{pmatrix}$, $\widetilde{W}_j=\begin{pmatrix}\widetilde{E}_j\\\widetilde{N}_j\end{pmatrix}(j=1, \cdots, J)$ be two solutions for equation $x=T_1(X)$. Let $\Psi_j=E_j-\widetilde{E}_j$, $\Phi_j=N_j-\widetilde{N}_j$, so that

$$i\Psi_j = -\frac{k}{h^2}\Delta_+\Delta_-\Psi_j + k\alpha_j(\Phi_j E_j + \widetilde{N}_j\Psi_j) - k\beta_j(|E_j|^2 E_j - |\widetilde{E}_j|^2\widetilde{E}_j), \tag{3.7}$$

$$\Phi_j = \frac{k^2}{h^2}\Delta_+\Delta_-\Phi_j - \gamma k\Phi_j + \frac{k^2}{h^2}\Delta_+\Delta_-(|E_j|^2 - |\widetilde{E}_j|^2), \quad j=1, \cdots, J. \tag{3.8}$$

It can be seen that

$$\Psi_j = \Psi_{j+J}, \quad \Phi_j = \Phi_{j+J}. \tag{3.9}$$

Multiplying (3.7) by $\overline{\Psi}_j$, summing up for $j=1, \cdots, J$ and then separating the imaginary part, we obtain

$$\begin{aligned}\sum_{j=1}^J |\Psi_j|^2 &\leqslant k\alpha^* \sum_{j=1}^J |\Phi_j||E_j||\Psi_j| + k\beta^* \sum_{j=1}^J (|E_j| + |\widetilde{E}_j|)|\Psi_j|^2 |E_j| \\ &\leqslant \frac{1}{2}k\alpha^* K\Big(\sum_{j=1}^J |\Phi_j|^2 + \sum_{j=1}^J |\Psi_j|^2\Big) + 2k\beta^* K^2 \sum_{j=1}^J |\Psi_j|^2,\end{aligned}$$

or

$$\sum_{j=1}^J |\Psi_j|^2 \leqslant \frac{\frac{1}{2}k\alpha^* K}{1 - k\Big(\frac{1}{2}\alpha^* K + 2\beta^* K^2\Big)} \sum_{j=1}^J |\Phi_j|^2. \tag{3.10}$$

Multiplying (3.8) by Φ_j, and summing for $j=1, \cdots, J$, from Lemma 1 and (3.9), we have

$$\sum_{j=1}^J |\Phi_j|^2 \leqslant -\frac{k^2}{h^2}\sum_{j=1}^J |\Delta_+\Phi_j|^2 - \gamma k \sum_{j=1}^J |\Phi_j|^2 + \frac{k^2}{h^2}\sum_{j=1}^J |\Delta_+\Phi_j||\Delta_+(|E_j|^2 - |\widetilde{E}_j|^2)|.$$

Using (3.6), we have $|\Delta_+(|E_j|^2 - |\widetilde{E}_j|^2)| \leqslant |(|E_{j+1}|^2 - |\widetilde{E}_{j+1}|^2)| + |(|E_j|^2 - |\widetilde{E}_j|^2)| \leqslant 2K(|\Psi_{j+1}| + |\Psi_j|)$. Hence

$$\begin{aligned}\sum_{j=1}^J |\Delta_+\Phi_j||\Delta_+(|E_j|^2 - |\widetilde{E}_j|^2)| &\leqslant \sum_{j=1}^J 2K|\Delta_+\Phi_j|(|\Psi_{j+1}| + |\Psi_j|) \\ &\leqslant \sum_{j=1}^J |\Delta_+\Phi_j|^2 + K^2 \sum_{j=1}^J (|\Psi_{j+1}| + |\Psi_j|)^2.\end{aligned}$$

Using $\frac{k}{h} \leqslant C$ and $\sum_{j=1}^J |\Psi_{j+1}|^2 = \sum_{j=1}^J |\Psi_j|^2$, we obtain

$$\sum_{j=1}^J |\Phi_j|^2 \leqslant -\gamma k \sum_{j=1}^J |\Phi_j|^2 + 4C^2 K^2 \sum_{j=1}^J |\Psi_j|^2.$$

Substituting (3.10) into the right-hand side of the above expression, we obtain

$$\left[1 - |\gamma| k - \frac{2k\alpha^* K^3 C^2}{1 - k\Big(\frac{1}{2}\alpha^* K + 2\beta^* K^2\Big)}\right] \sum_{j=1}^J |\Phi_j|^2 \leqslant 0.$$

When $1 - k\,(|\gamma| + \frac{1}{2}\alpha^* K + 2\beta^* K + 2\alpha^* K^3 C^2) > 0$, there is $\sum_{j=1}^J |\Phi_j|^2 = 0$. Using (3.10), we have $\sum_{j=1}^J |\Psi_j|^2 = 0$. The proof is completed.

4. Priori Estimation for Difference Solution

In this section we omit "h" from ε^h, η^h. The system (2.1)—(2.4) can be written in the form

$$i\varepsilon_t + \varepsilon_{xx} - \alpha(x)\eta\varepsilon + \beta(x)|\varepsilon|^2\varepsilon = 0; \tag{4.1}$$

$$\eta_{tt} - \eta_{xx} + \gamma\eta_t = |\varepsilon|^2_{xx}, \tag{4.2}$$

$$\varepsilon(x_j,\ t)=\varepsilon(x_{j+J},\ t),\ \eta(x_j,\ t)=\eta(x_{j+J},\ t),\quad \forall j,\ t\geqslant 0, \tag{4.3}$$

$$\varepsilon(x_j,\ 0)=\varepsilon_0(x_j),\ \eta(x_j,\ 0)=\eta_0(x_j),\quad \eta_{\bar t}(x_j,\ 0)=\eta_1(x_j). \tag{4.4}$$

Lemma 2. *Suppose $\varepsilon_0(x)\in C[0,\ D]$. For any $0\leqslant t\leqslant T$,*

$$\|\varepsilon\|_h^2\leqslant 2\|\varepsilon_0\|_{L_2}^2=E_0^2,\quad h<h_0,$$

where E_0 is independent of k and h.

Proof. We take the inner product of (4.1) with ε and then take the imaginary prart, thus obtaining the relation

$$\frac{1}{2}(\|\varepsilon\|_h^2)_{\bar t}+\frac{k}{2}\|\varepsilon_{\bar t}\|_h^2=0.$$

It follows that

$$\|\varepsilon\|_h^2\leqslant\|\varepsilon_0\|_h^2\leqslant 2\|\varepsilon_0\|_{L_2}^2=E_0^2,\quad h<h_0.$$

Lemma 3 (Sobolev's ineqality for difference operator)[10]. *Given $\sigma>0$, there exists $\zeta>0$, such that*

$$\left\|\frac{\Delta_+^s\phi}{h^s}\right\|_h\leqslant\zeta\|\phi\|_h+\sigma\left\|\frac{\Delta_+^n\phi}{h^n}\right\|_h,\quad s\leqslant n,$$

$$\left\|\frac{\Delta_+^s\phi}{h^s}\right\|_{L_\infty}\leqslant\zeta\|\phi\|_h+\sigma\left\|\frac{\Delta_+^n\phi}{h^n}\right\|_h,\quad s<n.$$

Lemma 4. *Let $u(t)$ be a nonnegative function satisfying the following conditions:*

$$u_{\bar t}\leqslant C_1u^q+C_2,\quad q>0,$$

$$u|_{t=0}=u_0.$$

Then there exists a constant T_0, such that for $0\leqslant t\leqslant T_0$,

$$u(t)\leqslant C_3$$

holds, where C_3 is dependent on C_1, C_2, q, u_0 and T_0.

Lemma 5. *Suppose that the following conditions are satisfied*

(1) $\varepsilon_0(x)\in C^2[0,\ D]$,

(2) $\eta_0(x)\in C^1[0,\ D]$, $\eta_1(x)\in C[0,\ D]$,

(3) $\max\limits_{0<x<D}|\alpha(x)|\leqslant\alpha^*$, $\max\limits_{0<x<D}|\beta(x)|\leqslant\beta^*$, $\gamma>0$,

(4) $\frac{k}{h}<C$, *C is an arbitrarily given positive constant,*

then there exsist a T_0, such that for $0\leqslant t\leqslant T_0$,

$$\|\varepsilon_{\bar t}\|_h^2+\|\varepsilon_{\bar x}\|_h^2+\|\eta\|_h^2+\|\eta_{\bar t}\|_h^2+\|\eta_{\bar x}\|_h^2\leqslant E_1^2,\quad h<h_0.$$

Proof. (1) Making the difference quotient of (4.1) with respect to t, and then making the inner product of the resulting relation with $\varepsilon_{\bar t}$ and separating the imaginary part, we obtain an expression. Note that the terms of the right-hand side of the expression has the following estimations, respectively

$$\sum|\eta|\,|\varepsilon\cdot\bar\varepsilon_{\bar t}|h\leqslant C_1+C_2(\|\varepsilon_{\bar x}\|_h^8+\|\eta_{\bar t}\|_h^8+\|\varepsilon_{\bar t}\|_h^8),$$

$$\sum|\varepsilon\cdot\varepsilon_{\bar t}|^2h\leqslant C_3+C_4(\|\varepsilon_{\bar x}\|_h^8+\|\varepsilon_{\bar t}\|_h^8),$$

$$k\beta^* \sum_{j=1}^{J} |\varepsilon_{\bar{t}}|^2 |\varepsilon \cdot \bar{\varepsilon}_{\bar{t}}| h \leqslant C_5 + C_6(\|\varepsilon_{\bar{x}}\|_h^8 + \|\varepsilon_{\bar{t}}\|_h^8,$$

where we have applied Lemmas 2, 3 and $\frac{k}{h} \leqslant C$. Thus

$$(\|\varepsilon_{\bar{t}}\|_h^2)_{\bar{t}} \leqslant C_7 + C_8(\|\varepsilon_{\bar{x}}\|_h^8 + \|\varepsilon_{\bar{t}}\|_h^8 + \|\eta_{\bar{t}}\|_h^8). \tag{4.5}$$

(2) We take the inner product of (4.1) with $\varepsilon_{x\bar{x}}$ and take the imaginary part to obtain the inequality

$$\frac{1}{2}(\|\varepsilon_{\bar{x}}\|_h^2)_{\bar{t}} \leqslant \alpha^* \sum_{j=1}^{J} |\eta| |\varepsilon \cdot \bar{\varepsilon}_{x\bar{x}}| + \beta^* \sum_{j=1}^{J} |\varepsilon|^2 |\varepsilon \cdot \bar{\varepsilon}_{x\bar{x}}| h$$
$$\leqslant C_9 + C_{10}(\|\varepsilon_{\bar{x}}\|_h^8 + \|\eta\|_h^8 + \|\varepsilon_{x\bar{x}}\|_h^4). \tag{4.6}$$

From (4.1) we get

$$\|\varepsilon_{x\bar{x}}\|_h \leqslant C_{11} + C_{12}(\|\varepsilon_{\bar{t}}\|_h^2 + \|\varepsilon_{\bar{x}}\|_h^2 + \|\eta\|_h^2). \tag{4.7}$$

Substituting (4.7) into (4.6), we have

$$(\|\varepsilon_{\bar{x}}\|_h^2)_{\bar{t}} \leqslant C_{13} + C_{14}(\|\varepsilon_{\bar{t}}\|_h^8 + \|\varepsilon_{\bar{x}}\|_h^8 + \|\eta\|_h^8). \tag{4.8}$$

(3) We take the inner product of (4.2) with $\eta_{\bar{t}}$ to obtain the inequality

$$\frac{1}{2}(\|\eta_{\bar{t}}\|_h^2 + \|\eta_{\bar{x}}\|_h^2)_{\bar{t}} \leqslant \sum_{j=1}^{J} |\eta_{\bar{t}}| |\varepsilon|_{x\bar{x}}^2 h.$$

Since $|\varepsilon|_{x\bar{x}}^2 = |\varepsilon_x|^2 + |\varepsilon_{\bar{x}}|^2 + \bar{\varepsilon} \cdot \varepsilon_{x\bar{x}} + \varepsilon \cdot \bar{\varepsilon}_{x\bar{x}}$, it follows that

$$\frac{1}{2}(\|\eta_{\bar{t}}\|_h^2 + \|\eta_{\bar{x}}\|_h^2)_{\bar{t}} \leqslant C_{15} + C_{16}(\|\varepsilon_{\bar{x}}\|_h^8 + \|\eta_{\bar{t}}\|_h^8 + \|\varepsilon_{x\bar{x}}\|_h^4). \tag{4.9}$$

Substituting (4.7) into (4.9), we get

$$(\|\eta_{\bar{t}}\|_h^2 + \|\eta_{\bar{x}}\|_h^2)_{\bar{t}} \leqslant C_{17} + C_{18}(\|\varepsilon_{\bar{t}}\|_h^8 + \|\varepsilon_{\bar{x}}\|_h^8 + \|\eta\|_h^8 + \|\eta_{\bar{t}}\|_h^8). \tag{4.10}$$

(4) $$(\eta^2)_{\bar{t}} = [\eta(t) + \eta(t-k)]\eta_{\bar{t}} \leqslant \frac{1}{2}[\eta(t) + \eta(t-k)]^2 + \frac{1}{2}(\eta_{\bar{t}})^2$$
$$\leqslant \eta^2(t) + \eta^2(t-k) + \frac{1}{2}(\eta_{\bar{t}})^2.$$

Hence $(\eta^2)_{\bar{t}} + k(\eta^2)_{\bar{t}} \leqslant 2\eta^2 + \frac{1}{2}(\eta_{\bar{t}})^2$, or $(\eta^2)_{\bar{t}} \leqslant \frac{1}{1+k}\left(2\eta^2 + \frac{1}{2}(\eta_{\bar{t}})^2\right)$. Now

$$(\|\eta\|_h^2)_{\bar{t}} \leqslant C_{19} + C_{20}(\|\eta\|_h^2 + \|\eta_{\bar{t}}\|_h^2). \tag{4.11}$$

Combining (4.5), (4.8), (4.10), (4.11), we have

$$(\|\varepsilon_{\bar{t}}\|_h^2 + \|\varepsilon_{\bar{x}}\|_h^2 + \|\eta\|_h^2 + \|\eta_{\bar{t}}\|_h^2 + \|\eta_{\bar{x}}\|_h^2)_{\bar{t}} \leqslant C_{21} + C_{22}(\|\varepsilon_{\bar{t}}\|_h^2 + \|\varepsilon_{\bar{x}}\|_h^2 + \|\eta_{\bar{t}}\|_h^2 + \|\eta\|_h^2 + \|\eta_{\bar{x}}\|_h^2)^4.$$

Finally, it follows from Lemma 4 that

$$\|\varepsilon_{\bar{t}}\|_h^2 + \|\varepsilon_{\bar{x}}\|_h^2 + \|\eta\|_h^2 + \|\eta_{\bar{t}}\|_h^2 + \|\eta_{\bar{x}}\|_h^2 \leqslant E_1^2.$$

Deduction 1. Suppose the conditions for Lemma 5 are satisfied, then we have

$$\|\varepsilon_{x\bar{x}}\|_h \leqslant E_2, \quad 0 \leqslant t \leqslant T_0.$$

Deduction 2. Suppose the conditions for Lemma 5 are satisfied, then we have

$$\|\varepsilon\|_{L_\infty} \leqslant E_3, \quad \|\eta\|_{L_\infty} \leqslant E_4, \quad \|\varepsilon_{\bar{x}}\|_{L_\infty} \leqslant E_5, \quad 0 \leqslant t \leqslant T_0.$$

Lemma 6. *Suppose that the conditions for Lemma 5 and the following conditions are satisfied*

(1) $\varepsilon_0(x)\in C^4[0, D]$,

(2) $\eta_0(x)\in C^2[0, D]$, $\eta_1(x)\in C^1[0, D]$.

Then we have

$$\|\varepsilon_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}\bar{t}}\|_h^2\leqslant E_6^2,\quad 0\leqslant t\leqslant T_0.$$

Proof. Making the difference quotient of (4.1) with respect to t, and then taking norm $\|\cdot\|_h$, we get

$$\|\varepsilon_{\bar{t}x\bar{x}}\|_h\leqslant\|\varepsilon_{\bar{t}\bar{t}}\|_h+\alpha^*\|(\eta\varepsilon)_{\bar{t}}\|_h+\beta^*\|(|\varepsilon|^2\varepsilon)_{\bar{t}}\|_h\leqslant C_{25}+\|\varepsilon_{\bar{t}\bar{t}}\|_h. \tag{4.12}$$

Making the difference quotient of (4.1) twice with respect to t and making the difference quotient (4.2) with respect to t, we obtain

$$i\varepsilon_{\bar{t}\bar{t}\bar{t}}+\varepsilon_{\bar{t}\bar{t}x\bar{x}}-\alpha(\eta\varepsilon)_{\bar{t}\bar{t}}+\beta(|\varepsilon|^2\varepsilon)_{\bar{t}\bar{t}}=0, \tag{4.13}$$

$$\eta_{\bar{t}\bar{t}\bar{t}}-\eta_{\bar{t}x\bar{x}}+\gamma\eta_{\bar{t}\bar{t}}=|\varepsilon|^2_{\bar{t}x\bar{x}}. \tag{4.14}$$

Taking the inner product of (4.14) with $\eta_{\bar{t}\bar{t}}$, we get

$$\frac{1}{2}(\|\eta_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}x}\|_h^2)_{\bar{t}}\leqslant\sum_{j=1}^{J}\big||\varepsilon|^2_{\bar{t}x\bar{x}}\big|\,|\eta_{\bar{t}\bar{t}}|h\leqslant\frac{1}{2}\,\||\varepsilon|^2_{\bar{t}x\bar{x}}\|_h^2+\frac{1}{2}\|\eta_{\bar{t}\bar{t}}\|_h^2.$$

Applying Lemma 3 and inequality (4.12), we obtain

$$\||\varepsilon|^2_{\bar{t}x\bar{x}}\|_h\leqslant C_{26}+C_{27}\|\varepsilon_{\bar{t}\bar{t}}\|_h.$$

Therefore

$$\frac{1}{2}(\|\eta_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}x}\|_h^2)_{\bar{t}}\leqslant C_{28}+C_{29}(\|\varepsilon_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}\bar{t}}\|_h^2). \tag{4.15}$$

Taking the inner product of (4.13) with $\varepsilon_{\bar{t}\bar{t}}$ and separating the imaginary part, we have

$$\frac{1}{2}(\|\varepsilon_{\bar{t}\bar{t}}\|_h^2)_{\bar{t}}\leqslant\alpha^*\sum_{j=1}^{J}|(\eta\varepsilon)_{\bar{t}\bar{t}}|\,|\varepsilon_{\bar{t}\bar{t}}|h+\beta^*\sum_{j=1}^{J}|(|\varepsilon|^2\varepsilon)_{\bar{t}\bar{t}}|\,|\varepsilon_{\bar{t}\bar{t}}|h$$
$$\leqslant C_{30}+C_{31}(\|\varepsilon_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}x}\|_h^2). \tag{4.16}$$

It follows from (4.15), (4.16) that

$$(\|\varepsilon_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}x}\|_h^2)_{\bar{t}}\leqslant C_{32}+C_{33}(\|\varepsilon_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}x}\|_h^2).$$

Applying Lemma 4, we obtain

$$\|\varepsilon_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}\bar{t}}\|_h^2+\|\eta_{\bar{t}x}\|_h^2\leqslant E_6^2,\quad 0\leqslant t\leqslant T_0.$$

Deduction 3. Suppose the conditions for Lemma 6 are satisfied, we have

$$\|\varepsilon_{\bar{t}x\bar{x}}\|_h\leqslant E_7,\ \|\eta_{x\bar{x}}\|_h\leqslant E_8,\quad 0\leqslant t\leqslant T_0.$$

Deduction 4. Suppose the conditions for Lemma 6 are satisfied, we have

$$\|\varepsilon_{\bar{t}}\|_{L_\infty}\leqslant E_9,\ \|\varepsilon_{x\bar{x}}\|_{L_\infty}\leqslant E_{10},\quad 0\leqslant t\leqslant T_0.$$

5. Convergence and Stability

Let ε^h, η^h denote the solution of problem $(2)^h$ and ε, η denote the solution of problem (1.1)—(1.4).

Theorem 6. *Suppose that ε, η are the classical smooth solution of problem (1.1)—(1.4), ε^h, η^h are the difference solution of problem $(2)^h$ and the following conditions are satisfied*

(1) $\varepsilon_0(x)\in C^4[0, D]$,

(2) $\eta_0(x)\in C^2[0, D]$, $\eta_1(x)\in C^1[0, D]$,

(3) $\max\limits_{0<x<D}|\alpha(x)|\leqslant\alpha^*$, $\max\limits_{0<x<D}|\beta(x)|\leqslant\beta^*$, $\gamma>0$,

(4) $\frac{k}{h}\leqslant C$, C *is any given gositive constant.*

Then the difference solution of problem $(2)^h$ *converges to the classical smooth solution of problem* (1.1)—(1.4) *and there exists the estimation*

$$\|\varepsilon-\varepsilon^h\|_h^2+\|\varepsilon_{\bar t}-\varepsilon^h_{\bar t}\|_h^2+\|\varepsilon_{\bar x}-\varepsilon^h_{\bar x}\|_h^2+\|\varepsilon_{x\bar x}-\varepsilon^h_{x\bar x}\|_h^2+\|\eta-\eta^h\|_h^2$$
$$+\|\eta_{\bar t}-\eta^h_{\bar t}\|_h^2+\|\eta_{\bar x}-\eta^h_{\bar x}\|_h^2\leqslant[O(k+h^2)]^2e^{CT_0}.$$

Proof. Substituting $\varepsilon(x, t)$, $\eta(x, t)$ into the difference equations (2.1), (2.2), we make the Taylor expansion to obtain

$$i\varepsilon_{\bar t}+\varepsilon_{x\bar x}-\alpha\eta\varepsilon+\beta|\varepsilon|^2\varepsilon=O(k+h^2), \tag{5.1}$$

$$\eta_{t\bar t}-\eta_{x\bar x}+\gamma\eta_{\bar t}=|\varepsilon|^2_{x\bar x}+O(k+h^2). \tag{5.2}$$

Let $\eta-\eta^h=\phi$, $\varepsilon-\varepsilon^h=\psi$. Then

$$i\psi_{\bar t}+\psi_{x\bar x}-\alpha(\eta\psi+\phi\varepsilon^h)+\beta(|\varepsilon|^2\varepsilon-|\varepsilon^h|^2\varepsilon^h)=O(k+h^2), \tag{5.3}$$

$$\phi_{t\bar t}-\phi_{x\bar x}+\gamma\phi_{\bar t}=|\varepsilon|^2_{x\bar x}-|\varepsilon^h|^2_{x\bar x}+O(k+h^2), \tag{5.4}$$

$$\phi(x_j, t)=\phi(x_{j+J}, t),\ \psi(x_j, t)=\psi(x_{j+J}, t),\quad \forall j,\ t\geqslant0, \tag{5.5}$$

$$\phi(x_j, 0)=0,\quad \psi(x_j, 0)=0. \tag{5.6}$$

(1) Making the inner product of (5.4) with $\phi_{\bar t}$, we have

$$\frac{1}{2}(\|\phi_{\bar t}\|_h^2+\|\phi_{\bar x}\|_h^2)_t\leqslant\sum_{j=1}^{J}|\phi_{\bar t}(|\varepsilon|^2_{x\bar x}-|\varepsilon^h|^2_{x\bar x})|h+\sum_{j=1}^{J}|\phi_{\bar t}O(k+h^2)|h.$$

It is easy to see that, from

$$||\varepsilon|^2_{x\bar x}-|\varepsilon^h|^2_{x\bar x}|\leqslant|\psi_x|(|\varepsilon_x|+|\varepsilon^h_x|)+|\psi_{\bar x}|(|\varepsilon_{\bar x}|+|\varepsilon^h_{\bar x}|)+2|\varepsilon||\psi_{x\bar x}|+2|\psi||\varepsilon^h_{x\bar x}|,$$

it follows that

$$\sum_{j=1}^{J}|\phi_{\bar t}(|\varepsilon|^2_{x\bar x}-|\varepsilon^h|^2_{x\bar x})|h\leqslant C_1(\|\phi_{\bar t}\|_h^2+\|\psi\|_h^2+\|\psi_{x\bar x}\|_h^2).$$

Here the uniformly bounded estimations $\|\varepsilon^h_{\bar x}\|_{L_\infty}$, $\|\varepsilon^h_{x\bar x}\|_{L_\infty}$ and $\|\varepsilon\|_{L_\infty}$, $\|\varepsilon_x\|_{L_\infty}$ are applied. In addition, from Lemma 3, $\|\psi_{\bar x}\|_{L_\infty}$ is replaced by $\|\psi\|_h$ and $\|\psi_{x\bar x}\|_h$. Since

$$\sum_{j=1}^{J}|\phi_{\bar t}O(k+h^2)|h\leqslant\frac{1}{2}\|\phi_{\bar t}\|_h^2+[O(k+h^2)]^2,$$

hence

$$(\|\phi_{\bar t}\|_h^2+\|\phi_{\bar x}\|_h^2)_t\leqslant C_2(\|\phi_{\bar t}\|_h^2+\|\psi\|_h^2+\|\psi_{x\bar x}\|_h^2)+[O(k+h^2)]^2. \tag{5.7}$$

By (5.3) it follows that

$$\|\psi_{x\bar x}\|_h\leqslant C_3(\|\psi\|_h+\|\psi_{\bar t}\|_h+\|\phi\|_h)+O(k+h^2). \tag{5.8}$$

Substituting (5.8) into (5.7), we have

$$(\|\phi_{\bar t}\|_h^2+\|\phi_{\bar x}\|_h^2)_t\leqslant C_4(\|\phi\|_h^2+\|\phi_{\bar t}\|_h^2+\|\psi\|_h^2+\|\psi_{\bar t}\|_h^2)+[O(k+h^2)]^2. \tag{5.9}$$

(2) Taking the inner product of (5.3) with ψ and separating the imaginary part, we obtain

$$\frac{1}{2}(\|\psi\|_h^2)_t\leqslant\alpha^*\sum_{j=1}^{J}|\phi\varepsilon^h\cdot\bar\psi|h+\beta^*\sum_{j=1}^{J}|(|\varepsilon|-|\varepsilon^h|)(|\varepsilon|+|\varepsilon^h|)\varepsilon^h\cdot\bar\psi|h+\sum_{j=1}^{J}|\psi\cdot O(k+h^2)|h$$
$$\leqslant C_5(\|\phi\|_h^2+\|\psi\|_h^2)+[O(k+h^2)]^2. \tag{5.10}$$

(3) In a way similar to that of (4) in Section 4, we get

$$(\|\phi\|_h^2)_{\bar{t}} \leqslant C_6(\|\phi\|_h^2 + \|\phi_{\bar{t}}\|_h^2). \tag{5.11}$$

(4) Making the difference quotient of (5.3) with respect to t, taking the inner product of the resulting relation with $\psi_{\bar{t}}$ and separating the imaginary part, we can obtain

$$\begin{aligned}\frac{1}{2}(\|\psi_{\bar{t}}\|_h^2)_{\bar{t}} &\leqslant \alpha^* \sum_{j=1}^{J} |(\eta\psi)_{\bar{t}} \cdot \bar{\psi}_{\bar{t}}| h + \alpha^* \sum_{j=1}^{J} |(\varepsilon^h\phi)_{\bar{t}} \cdot \bar{\psi}_{\bar{t}}| h + \beta^* \sum_{j=1}^{J} |(|\varepsilon|^2\psi)_{\bar{t}} \cdot \bar{\psi}_{\bar{t}}| h \\ &\quad + \beta^* \sum_{j=1}^{J} |[(|\varepsilon|^2 - |\varepsilon^h|^2)\varepsilon^h]_{\bar{t}} \cdot \bar{\psi}_{\bar{t}}| h \\ &\leqslant C_7(\|\phi\|_h^2 + \|\phi_{\bar{t}}\|_h^2 + \|\psi\|_h^2 + \|\psi_{\bar{t}}\|_h^2). \end{aligned} \tag{5.12}$$

Combining (5.9), (5.10), (5.11), (5.12), we have

$$\begin{aligned}&(\|\phi\|_h^2 + \|\phi_{\bar{t}}\|_h^2 + \|\phi_{\bar{x}}\|_h^2 + \|\psi\|_h^2 + \|\psi_{\bar{t}}\|_h^2)_{\bar{t}} \\ &\quad \leqslant C_8(\|\phi\|_h^2 + \|\phi_{\bar{t}}\|_h^2 + \|\phi_{\bar{x}}\|_h^2 + \|\psi\|_h^2 + \|\psi_{\bar{t}}\|_h^2 + [O(k+h^2)]^2.\end{aligned}$$

From Growall's inequality it follows that

$$\|\phi\|_h^2 + \|\phi_{\bar{t}}\|_h^2 + \|\phi_{\bar{x}}\|_h^2 + \|\psi\|_h^2 + \|\psi_{\bar{t}}\|_h^2 \leqslant [O(k+h^2)]^2 e^{CT_0}.$$

Combining the above result with (5.8) and Lemma 3, we obtain the conclution of Theorem 3.

Theorem 4. *Suppose the conditions of Theorem 3 are satisfied, then the difference solutions $\varepsilon^h(x, t)$, $\eta^h(x, t)$ are stable with norm $\|\cdot\|_{2,h}$ and $\|\cdot\|_{1,h}$ respectively.*

Proof. The proof is similar to that of Theorem 3.

The conditions (1), (2) of Theorems 3, 4 can be weakened to $\varepsilon_0(x) \in H^4(0, D)$, $\eta_0(x) \in H^2(0, D)$, $\eta_1(x) \in H^1(0, D)$. In fact, we can make $\{\varepsilon_0^s(x)\} \subset C^4[0, D]$, $\{\eta_0^s(x)\} \subset C^2[0, D]$, $\{\eta_1^s(x)\} \subset C^1[0, D]$ converge to $\varepsilon_0(x)$, $\eta_0(x)$, $\eta_1(x)$, contained in $H^4(0, D)$, $H^2(0, D)$, $H^1(0, D)$ respectively. Let ε^s, η^s be the difference solution corresponding to the initial values $\varepsilon_0^s(x)$, $\eta_0^s(x)$, $\eta_1^s(x)$. We can see that the estimations of Section 4 and Theorems 3, 4 of Section 5 for ε^s and η^s still hold. By the limit process with $s \to \infty$, we can obtain the same results as in Theorems 3, 4.

6. Numerical Experiments

We have made numerical computation of the scheme $(2)^h$ with the conditions: $\gamma = 0.1$, $\alpha(x) = 1$, $\beta(x) = 2$, $D = 4$, $k = 0.001$, $h = 0.04$, $\varepsilon(x, t) = (\varepsilon_1(x, t), \varepsilon_2(x, t))^T$.

(1) The first initial function was computed from

$$\varepsilon_1(x, 0) = \begin{cases} 0 & 0 \leqslant x < 1.4,\ 2.6 < x \leqslant 4, \\ 4+5i & 1.4 \leqslant x \leqslant 2.6; \end{cases}$$

$$\varepsilon_2(x, 0) = \begin{cases} 0 & 0 \leqslant x < 1.4,\ 2.6 < x \leqslant 4, \\ 2.5+3i & 1.4 \leqslant x \leqslant 2.6; \end{cases}$$

$$\eta(x, 0) = \begin{cases} 0 & 0 \leqslant x < 1.4,\ 2.6 < x \leqslant 4, \\ -40 & 1.4 \leqslant x \leqslant 2.6; \end{cases}$$

$$\eta_t(x, 0) = 0.$$

The numerical results are depicted in Fig. 1.

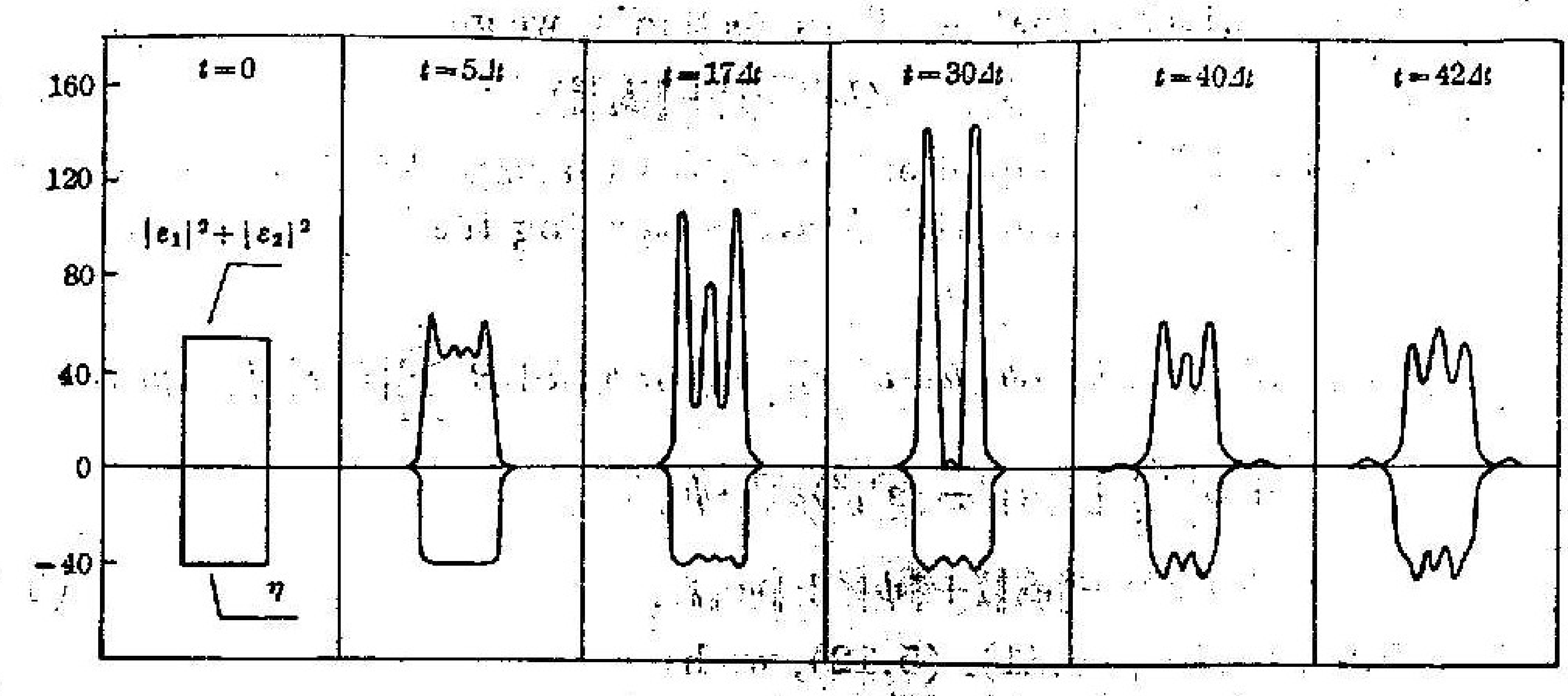

Fig. 1

(2) The second initial function was computed from

$$\varepsilon_1(x,\ 0)=4\sin^4\frac{\pi x}{4}+i5\sin^4\frac{\pi x}{4},$$

$$\varepsilon_2(x,\ 0)=3.5\sin^4\frac{\pi x}{4}+i3\sin^4\frac{\pi x}{4},$$

$$\eta(x,\ 0)=-120/(e^{-4x+8}+e^{4x-8})^2,$$

$$\eta_t(x,\ 0)=0.$$

The numerical results are depicted in Fig. 2.

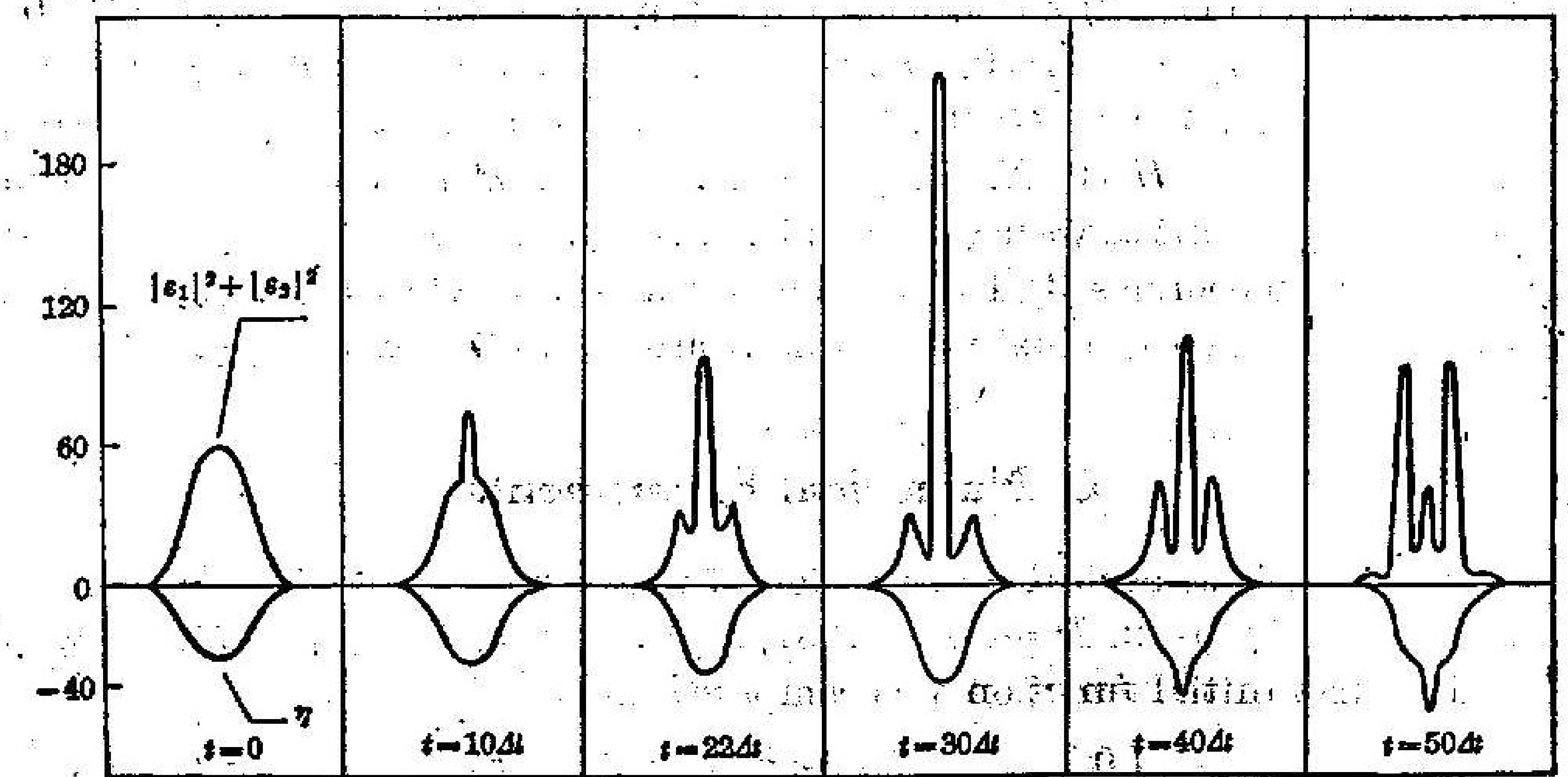

Fig. 2

References

[1] В. Е. Захаров, Коллапсленгмюровских Волн, Ж. Э. Т. Ф., Том. 62, 62. Вы. 5 (1972), 1745—1759.

[2] В. Е. Захаров, А. Ф. Мострюков, В. С. Сынах, Физика Плазмы, Том. 1, Вы. 4, 1975.

[3] G. J. Morales, Y. C. Lee, Generation of density cavities and localized electric fields in a nonuniform plasma, *Phys. Fluids*, Vol. 20, No. 7, 1977, 1135—1147.

[4] G. J. Morales, Y. C. Lee, Spiky turbulence generated by a propagating electrostatic wave of finite spatial extent, *Phys. Fluids*, Vol. 19, No. 5, 1976, 690—704.

[5] Guo Bo-ling, Shen Long-jun, The existence and uniqueness of the classical solution on the periodic initial value problem for Zakharov equation, *Acta Math. Appl. Sinica*, Vol. 5, No 3, 1982, 310—324.
[6] Guo Bo-ling, The convergence and stability of a class of finite difference scheme for periodic boundary problem of Zakharov equation, *Comp. Math. Sinica*, Vol. 4, 1982, 365—372.
[7] Чжоу Юй-линь, Краевые задачи для нелинейных параболических уравнений, *Математический сборник*, т. 47 (89), № 4, 1959, 431—484.
[8] A. Menikof, The existence of unbounded solution, *Comm. Pure Appl. Math.*, Vol. 25, 1972, 407—432.