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A PRIORI ERROR ESTIMATES OF A FINITE ELEMENT METHOD FOR DISTRIBUTED FLUX RECONSTRUCTION*

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Abstract

This paper is concerned with a priori error estimates of a finite element method for numerical reconstruction of some unknown distributed flux in an inverse heat conduction problem. More precisely, some unknown distributed Neumann data are to be recovered on the interior inaccessible boundary using Dirichlet measurement data on the outer accessible boundary. The main contribution in this work is to establish the some a priori error estimates in terms of the mesh size in the domain and on the accessible/inaccessible boundaries, respectively, for both the temperature u and the adjoint state p under the lowest regularity assumption. It is revealed that the lower bounds of the convergence rates depend on the geometry of the domain. These a priori error estimates are of immense interest by themselves and pave the way for proving the convergence analysis of adaptive techniques applied to a general classes of inverse heat conduction problems. Numerical experiments are presented to verify our theoretical prediction.

Mathematics subject classification: 35R30, 65N30, 65N15.
Key words: Distributed flux, Inverse heat problems, Finite element method, Error estimates.

1. Introduction

Inverse heat conduction problems are frequently encountered in engineering and industrial applications. In this paper we address a priori error estimates of a finite element method for numerical reconstruction of some unknown distributed flux in an inverse heat conduction problem. More precisely, the unknown distributed Neumann data, called *fluxes* in the sequel, are to be determined on the interior inaccessible boundary using Dirichlet measurement data on the outer accessible boundary.

The flux distribution is of paramount practical interest in heat conduction processes, e.g., the real-time monitoring in steel industry [1], the visualization by liquid crystal thermography [9], and estimating the freezing front velocity in the solidification process [24]. But its accurate distribution is rather difficult to obtain on some inaccessible boundary, such as the interior

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boundary of nuclear reactors and steel furnaces. Engineers seek to estimate them from accessible outer boundary measurements, which naturally gives rise to the inverse problem of estimating the distribution of fluxes. The most difficult issue in solving and analyzing the inverse heat problem of recovering the distributed flux lies in the strong instability with respect to the errors in the measurement data, i.e., small perturbation in the measurement data may lead to significant amplification of error in the identified flux. It is well-known that the inverse problem under investigation here is essentially lack of continuous dependance on data, thus ill-posed in Hadamard's sense [12].

In order to achieve a reasonable and practically acceptable numerical reconstruction of the flux, one may have to resort to some regularization techniques to transform the unstable ill-posed heat flux reconstruction process into a stable mathematical one. Several numerical methods have been proposed for the distributed flux reconstruction problem, among which the least-squares formulation [23–25] has received intensive investigations and it has been implemented by means of the boundary integral method [25] and finite element method [23]. Recently, adaptive techniques are introduced in this field for efficiency consideration [16], which, guided by the a posteriori error estimates, refines automatically the mesh to better approximate the local but potentially very important features of the distributed flux, e.g., non-smooth boundaries, discontinuous fluxes, or singular fluxes with spikes or abrupt sign changes. The computational cost is significantly reduced since fine resolution is only necessary in the place that local features lie in. Subsequently, the convergence analysis of the adaptive algorithm is established in [17], which requires an important a priori error estimates to develop an estimate for the quasi-orthogonality of the the discretization error with respect to the energy norm, which explain the coupling relation of errors on two successive meshes.

In this work, we will fill in the gap aforementioned by establishing some important a priori error estimates of finite element solutions to the heat flux reconstruction problems, which are of immense interest in numerical analysis of FEMs. Furthermore, they pave the way for proving the convergence analysis of adaptive techniques applied to a general class of inverse heat conduction problems. The detailed convergence analysis is reported in a separate work [17]. Here we derive the convergence order by using the piecewise linear continuous finite elements in terms of the mesh size by assuming the least regularity of solutions to the PDE system associated with the inverse problem, which is of practical use for reconstructing distributed fluxes of salient features.

The paper is organized as follows. In Section 2, we briefly recall the mathematical description of our flux reconstruction problem by an output least-squares formulation plus some Tikhonov regularization term. Some relevant properties are shortly recalled without proof. In Section 3, the finite element discretization is described in detail for purpose of analysis. In Sections 4 and 5, we derive the a priori energy and L^2 norm error estimates in detail, respectively, under the least assumption of regularity. In Section 6, numerical results of two-dimensional problems on a square and an L-shaped domain are presented to demonstrate the theoretical convergence order from analysis. We conclude the work in Section 7 and point out some future work.

We end this section with some notations and conventions. Throughout the paper we adopt the standard notation $W^{m,p}(D)$ for Sobolev spaces on an open bounded domain D in $\mathbb{R}^d$, and write $H^m(D) = W^{m,2}(D)$ for $p = 2$. The norm and semi-norm of $H^m(D)$ are denoted respectively by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$. We use $(\cdot,\cdot)_D$ to denote the inner product in $L^2(D)$. When no confusion is caused, we may simply drop D in the notation $\|\cdot\|_{m,D}$ and $(\cdot,\cdot)_D$. In addition, we will often use c or C to denote generic positive constants which are independent of mesh size h and functions involved.

2. Mathematical Formulation

Let Ω be an open and bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with some smooth boundary Γ consisting of two disjointed parts, namely $\Gamma = \Gamma_a \cup \Gamma_i$. The boundaries Γ_i and Γ_a refer, respectively, to the part of the boundary Γ that is inaccessible or accessible to experimental measurement devices. The steady-state heat conduction problem could be described by the elliptic PDE:

$$\begin{cases} -\nabla \cdot (\alpha(x)\nabla u(x)) = f(x), & x \in \Omega, \\ \alpha(x)\dfrac{\partial u}{\partial n} + k(u(x) - u_a(x)) = 0, & x \in \Gamma_a, \\ \alpha(x)\dfrac{\partial u}{\partial n} + q(x) = 0, & x \in \Gamma_i, \end{cases} \tag{2.1}$$

where the given data include the heat source f, the ambient temperature u_a, the heat transfer (Robin) coefficient k and the heat conductivity α. The distribution of the Neumann data, or flux, $q(x)$ on Γ_i is the quantity of interest in this work.

The inverse problem that we are concerned with is to recover the distributed flux $q(x)$ on the interior inaccessible part Γ_i, given the partial measurement data $z(x)$ of temperature $u(x)$ on the outer accessible part Γ_a.

Due to the severe ill-posedness (see, e.g., [23, Theorem 2.2]), the reconstruction is carried out through the output least-squares formulation combined with an Tikhonov regularization term to determine $q(x)$ by minimizing the stabilized cost functional

$$J(q) = \frac{1}{2}\|u(q) - z\|^2_{0,\Gamma_a} + \frac{\beta}{2}\|q\|^2_{0,\Gamma_i} \tag{2.2}$$

over $q \in L^2(\Gamma_i)$, where β is the regularization parameter. Here $u(q) : L^2(\Gamma_i) \to H^1(\Omega)$ represent the solution operator of the direct problem (2.1), which maps the parameter q to the solution u to the PDE (2.1).

Following [23, Theorem 2.2], one can see that the reconstruction process of the distributed flux q is stabilized in the sense that the solution to (2.2) is stable with respect to the perturbation of noisy data. By the linear dependence of u on q, (2.2) can be viewed as a convex quadratic functional over the infinite linear space $L^2(\Gamma_i)$, which immediately implies the existence and uniqueness of the stabilized solution $q^* \in L^2(\Gamma_i)$.

The necessary and sufficient optimality conditions of the regularized formulation (2.2) are characterized by the following theorem (see [16, Theorem 2.1] and its associated proof).

Lemma 2.1. *The optimization problem* (2.2) *admits a unique solution q. Moreover, q is the minimizer if and only if there is a costate $p \in H^1(\Omega)$ such that the triplet (u, p, q) satisfies the following optimality conditions:*

$$\begin{cases} (\alpha\nabla u, \nabla\phi) + (ku, \phi)_{\Gamma_a} = (f, \phi) + (ku_a, \phi)_{\Gamma_a} - (q, \phi)_{\Gamma_i}, & \forall\phi \in H^1(\Omega), \\ (\alpha\nabla p, \nabla v) + (kp, v)_{\Gamma_a} = (u - z, v)_{\Gamma_a}, & \forall v \in H^1(\Omega), \\ J'(q)(w) = (\beta q - p, w)_{\Gamma_i} = 0, & \forall w \in L^2(\Gamma_i). \end{cases} \tag{2.3}$$

For later analysis, we define the energy norm by

$$|||\cdot|||_1^2 = (\alpha\nabla\cdot, \nabla\cdot) + (k\cdot, \cdot)_{\Gamma_a},$$

which is obviously equivalent to H^1-norm $\|\cdot\|_1$ due to Friedrich's inequality.

3. Finite Element Discretization

In the next few sections we will investigate the finite element approximations of the Tikhonov regularization system (2.1)–(2.2) formulated in Section 2 and their a priori error estimates in terms of the mesh size. For purpose of preparation, we introduce in this section a triangulation of the domain and some finite element spaces associated with the triangulation. We triangulate the polyhedral domain using a quasi-uniform mesh T^h consisting of simplicial elements of mesh size h (see [6]) such that $\overline{\Omega} = \cup_{\tau\in T^h}\bar{\tau}$. Associated with T^h is the continuous piecewise linear finite element subspace V^h of $C(\bar{\Omega})$:

$$V^h = \Big\{ v_h \in H^1(\Omega) \mid v_h|_\tau \in P_1(\tau),\ \forall\, \tau \in T^h \Big\},$$

which is used for the spacial discretization in the next few sections, where $P_1(\tau)$ is the space of polynomials of degree one in an element τ.

The natural restriction of T^h on the boundary of Ω forms the triangulations of Γ_i and Γ_a, denoted by Γ_i^h and Γ_a^h, respectively. Let F^h be the set of all faces of the triangulation T^h and F_0^h be the set of all faces which are not on the boundary of Ω, namely, $F^h = F_0^h \bigcup(\Gamma_i^h \cup \Gamma_a^h)$. Let h_τ denote the diameter of the element τ in T^h, and h_l the diameter of the edge l in ∂T^h. Then we take the feasible approximation space for fluxes q to be the natural restriction of V^h on the boundary Γ_i, denoted by $V^h_{\Gamma_i}$.

Then the discrete counterpart of the continuous problem (2.2) can be formulated as:

$$\min_{q_h\in V^h_{\Gamma_i}} J_h(q_h) = \frac{1}{2}\|u_h(q_h) - z\|^2_{0,\Gamma_a} + \frac{\beta}{2}\|q_h\|^2_{0,\Gamma_i}, \tag{3.1}$$

where $u_h(q_h) \in V^h$ is the finite element discretization of (2.1), whose variational form reads as: Seek $u_h \in V^h$ such that

$$(\alpha\nabla u_h, \nabla\phi_h) + (ku_h, \phi_h)_{\Gamma_a} = (f, \phi_h) + (ku_a, \phi_h)_{\Gamma_a} - (q_h, \phi_h)_{\Gamma_i}, \quad \forall \phi_h \in V^h. \tag{3.2}$$

As in Theorem 2.1, the discrete optimality conditions can be obtained by simply replacing (u, p, q) with (u_h, p_h, q_h) and continuous spaces with finite element spaces, respectively (cf. [16, Eq. (2.6)]).

$$\begin{cases} (\alpha\nabla u_h, \nabla\phi_h) + (ku_h, \phi_h)_{\Gamma_a} = (f, \phi_h) + (ku_a, \phi_h)_{\Gamma_a} - (q_h, \phi_h)_{\Gamma_i}, & \forall \phi_h \in V^h, \\ (\alpha\nabla p_h, \nabla\phi_h) + (kp_h, \phi_h)_{\Gamma_a} = (u_h - z, \phi_h)_{\Gamma_a}, & \forall \phi_h \in V^h, \\ J_h'(q_h)(w_h) = (\beta q_h - p_h, w_h)_{\Gamma_i} = 0, & \forall w_h \in V^h_{\Gamma_i}. \end{cases} \tag{3.3}$$

Before ending this section, let us recall some classical interpolation error estimates, which will be frequently used in the analysis afterward. For a given finite element space V^h, let I_h be the quasi-interpolation operator introduced in [7,22], which is well-posed even for the $L^2(\Omega)$ function and has the following error estimate:

$$\|v - I_h v\|_{L^2(\Omega)} + h\|v - I_h v\|_{H^1(\Omega)} \le Ch^s\|v\|_{H^s(\Omega)}, \quad \forall\, v \in H^s(\Omega)\,, \tag{3.4}$$

for any $s \in (1, 2]$. Some other projection operator like the one defined in [4, Cor. 5.3] is another proper candidate by obtaining the H^1 estimate in light of the commuting diagram property.

4. A Priori Energy Norm Error Estimate

In this section, we first derive the a priori energy norm error estimate of the finite element approximation of the inverse problem. The key issue lies in the L^2-norm error estimates of both u and p on the boundary.

The following lemma concerning the relationship between the L^2 error estimate on the boundary and the energy norm error estimate of both u and p, which will play a key role in the subsequent analysis.

Lemma 4.1. *Let (u,p,q) and (u_h,p_h,q_h) be the solutions of (2.3) and (3.3), respectively. Then there exists a constant $C>0$ depending only on the minimum angle of the mesh such that*

$$\|u-u_h\|_{0,\Gamma_a}^2+\|p-p_h\|_{0,\Gamma_i}^2 \le Ch^{\gamma}\Big(|||u-u_h|||_1^2+|||p-p_h|||_1^2\Big), \tag{4.1}$$

with the constant $\gamma\in(0,1]$ depending on the geometry of the domain.

Proof. Let $(u(q_h),p(q_h))$ be the solution pair of two auxiliary PDEs

$$\big(\alpha\nabla u(q_h),\nabla v\big)+\big(ku(q_h),v\big)_{\Gamma_a}=(f,v)+(ku_a,v)_{\Gamma_a}-(q_h,v)_{\Gamma_i},\quad \forall v\in H^1(\Omega), \tag{4.2}$$

$$\big(\alpha\nabla p(q_h),\nabla v\big)+\big(kp(q_h),v\big)_{\Gamma_a}=\big(u(q_h)-z,v\big)_{\Gamma_a},\quad \forall v\in H^1(\Omega). \tag{4.3}$$

First of all, since u_h can be regarded as a Galerkin solution of the elliptic problem (4.2), it follows from [6] the standard L^2 error estimate

$$\|u(q_h)-u_h\|_0\le Ch^{\gamma_1}|||u(q_h)-u_h|||_1, \tag{4.4}$$

with a constant $\gamma_1\in(0,1]$ depending on the geometry of the domain. Next, we have by the trace inequality

$$\begin{aligned}
&\|u(q_h)-u_h\|_{0,\Gamma_a}^2\\
\le\ &\|u(q_h)-u_h\|_{0,\Gamma}^2\\
\le\ &c_1\|u(q_h)-u_h\|_0\|u(q_h)-u_h\|_1\le Ch^{\gamma_1}|||u(q_h)-u_h|||_1^2,
\end{aligned} \tag{4.5}$$

from which we infer that

$$\begin{aligned}
&\|u-u_h\|_{0,\Gamma_a}^2\\
\le\ &2\|u-u(q_h)\|_{0,\Gamma_a}^2+2\|u(q_h)-u_h\|_{0,\Gamma_a}^2\\
\le\ &2\|u-u(q_h)\|_{0,\Gamma_a}^2+Ch^{\gamma_1}|||u(q_h)-u_h|||_1^2\\
\le\ &2(1+Ch^{\gamma_1})\|u-u(q_h)\|_{0,\Gamma_a}^2+Ch^{\gamma_1}|||u-u_h|||_1^2.
\end{aligned} \tag{4.6}$$

It remains to estimate the term $\|u-u(q_h)\|_{0,\Gamma_a}^2$. By subtracting (4.2) and (4.3) from (2.3), respectively, we have

$$(\alpha\nabla\big(u(q_h)-u\big),\nabla v)+(k\big(u(q_h)-u\big),v)_{\Gamma_a}=(q-q_h,v)_{\Gamma_i},\quad \forall v\in H^1(\Omega), \tag{4.7}$$

$$(\alpha\nabla\big(p(q_h)-p\big),\nabla w)+(k\big(p(q_h)-p\big),w)_{\Gamma_a}=(u(q_h)-u,w)_{\Gamma_a},\quad \forall v\in H^1(\Omega). \tag{4.8}$$

Then by taking $v=p(q_h)-p$ in (4.7) and $w=u(q_h)-u$ in (4.8), we obtain

$$\begin{aligned}
&\|u(q_h)-u\|_{0,\Gamma_a}^2\\
=\ &(q-q_h,p(q_h)-p)_{\Gamma_i}=\beta^{-1}(p-p_h,p(q_h)-p)_{\Gamma_i}\\
=\ &-\beta^{-1}\|p(q_h)-p\|_{0,\Gamma_i}^2+\beta^{-1}(p(q_h)-p_h,p(q_h)-p)_{\Gamma_i},
\end{aligned} \tag{4.9}$$

which implies directly that

$$\|u(q_h) - u\|_{0,\Gamma_a}^2 + \frac{1}{2\beta}\|p(q_h) - p\|_{0,\Gamma_i}^2 \le \frac{1}{2\beta}\|p(q_h) - p_h\|_{0,\Gamma_i}^2. \tag{4.10}$$

To bound $\|p(q_h) - p_h\|_{0,\Gamma_i}$, let us introduce another auxiliary PDE:

$$\begin{cases} -\nabla\cdot(\alpha\nabla\phi) = 0, & \text{in } \Omega, \\ \alpha\dfrac{\partial\phi}{\partial n} + k\phi = 0, & \text{on } \Gamma_a, \\ \alpha\dfrac{\partial\phi}{\partial n} = p(q_h) - p_h, & \text{on } \Gamma_i. \end{cases} \tag{4.11}$$

It follows from [6] that there exists a constant $\gamma_2 \in (0, \frac{1}{2}]$ depending on the geometry of the domain such that the above problem has the following regularity result

$$\|\phi\|_{1+\gamma_2} \le C\||p(q_h) - p_h|\|_{0,\Gamma_i}, \tag{4.12}$$

with its variational problem reads as: Seek $\phi \in H^1(\Omega)$

$$(\alpha\nabla\phi, \nabla v) + (k\phi, v)_{\Gamma_a} = (p(q_h) - p_h, v)_{\Gamma_i}, \quad \forall v \in H^1(\Omega). \tag{4.13}$$

We put $v = p(q_h) - p_h$ in (4.13) then by (4.3) and the second equation in (3.3),

$$\begin{aligned}
&\|p(q_h) - p_h\|_{0,\Gamma_i}^2 \\
&= \big(\alpha\nabla\phi, \nabla p(q_h)\big) + \big(k\phi, p(q_h)\big)_{\Gamma_a} - (\alpha\nabla\phi, \nabla p_h) - (k\phi, p_h)_{\Gamma_a} \\
&= \big(u(q_h) - z, \phi\big)_{\Gamma_a} - (u_h - z, I_h\phi)_{\Gamma_a} - \big(\alpha\nabla(\phi - I_h\phi), \nabla p_h\big) - \big(k(\phi - I_h\phi), p_h\big)_{\Gamma_a} \\
&= \big(u(q_h) - u_h, \phi\big)_{\Gamma_a} + \big(u_h - z, \phi - I_h\phi\big)_{\Gamma_a} - \big(\alpha\nabla(\phi - I_h\phi), \nabla p_h\big) - \big(k(\phi - I_h\phi), p_h\big)_{\Gamma_a} \\
&= \big(u(q_h) - u_h, \phi\big)_{\Gamma_a} + \big(u_h - z, \phi - I_h\phi\big)_{\Gamma_a} + \big(\alpha\nabla(\phi - I_h\phi), \nabla(p - p_h)\big) \\
&\quad + \big(k(\phi - I_h\phi), p - p_h\big)_{\Gamma_a} - \big(\alpha\nabla(\phi - I_h\phi), \nabla p\big) - \big(k(\phi - I_h\phi), p\big)_{\Gamma_a} \\
&= \big(u(q_h) - u_h, \phi\big)_{\Gamma_a} + \big(u_h - u, \phi - I_h\phi\big)_{\Gamma_a} \\
&\quad + \big(\alpha\nabla(\phi - I_h\phi), \nabla(p - p_h)\big) + \big(k(\phi - I_h\phi), p - p_h\big)_{\Gamma_a} \\
&=: A_1 + A_2 + A_3 + A_4.
\end{aligned} \tag{4.14}$$

In the following, we estimate the A_i's ($i = 1, 2, 3, 4$) term by term. Firstly, by the Cauchy-Schwarz inequality, the trace inequality, the estimates (4.5) and (4.12), we derive

$$\begin{aligned}
|A_1| &= \Big|\big(u(q_h) - u_h, \phi\big)_{\Gamma_a}\Big| \le \|u(q_h) - u_h\|_{0,\Gamma_a}\|\phi\|_{0,\Gamma} \\
&\le Ch^{\frac{\gamma_1}{2}}\||u(q_h) - u_h|\|_1\|\phi\|_{1+\gamma_2} \\
&\le Ch^{\frac{\gamma_1}{2}}\||u(q_h) - u_h|\|_1\|p(q_h) - p_h\|_{0,\Gamma_i} \\
&\le Ch^{\frac{\gamma_1}{2}}\Big(\||u(q_h) - u|\|_1 + \||u - u_h|\|_1\Big)\|p(q_h) - p_h\|_{0,\Gamma_i}.
\end{aligned} \tag{4.15}$$

The bound of the term $\||u(q_h) - u|\|_1$ can be obtained by setting $v = u(q_h) - u$ in (4.7),

$$\begin{aligned}
&\||u(q_h) - u|\|_1^2 \\
&= \big(q - q_h, u(q_h) - u\big)_{\Gamma_i} \\
&\le C\|q - q_h\|_{0,\Gamma_i}\|u(q_h) - u\|_{0,\Gamma_i} \le C\|q - q_h\|_{0,\Gamma_i}\||u(q_h) - u|\|_1,
\end{aligned} \tag{4.16}$$

then we can infer from the above two inequalities (4.16) and (4.15) that

$$|A_1| \le Ch^{\frac{\gamma_1}{2}}\Big(\|q-q_h\|_{0,\Gamma_i}+|||u-u_h|||_1\Big)\|p(q_h)-p_h\|_{0,\Gamma_i}. \tag{4.17}$$

By the Cauchy-Schwarz inequality, the trace inequality, the interpolation error estimate of the linear operator (3.4) and (4.12), A_2 can be bounded as follows:

$$\begin{aligned}|A_2| &= \Big|\big(u_h-u,\phi-I_h\phi\big)_{\Gamma_a}\Big| \le \|u-u_h\|_{0,\Gamma_a}\|\phi-I_h\phi\|_{0,\Gamma}\\ &\le C\|u-u_h\|_{0,\Gamma_a}\|\phi-I_h\phi\|_0^{\frac12}\|\phi-I_h\phi\|_1^{\frac12}\\ &\le Ch^{\frac12+\gamma_2}\|u-u_h\|_{0,\Gamma_a}\|\phi\|_{1+\gamma_2}\\ &\le Ch^{\frac12+\gamma_2}|||u-u_h|||_1\|p(q_h)-p_h\|_{0,\Gamma_i}.\end{aligned} \tag{4.18}$$

Likewise, A_3 can be estimated by

$$\begin{aligned}|A_3| &= \Big|\big(\alpha\nabla(\phi-I_h\phi),\nabla(p-p_h)\big)\Big|\\ &\le Ch^{\frac12+\gamma_2}\|p-p_h\|_1\|p(q_h)-p_h\|_{0,\Gamma_i}.\end{aligned} \tag{4.19}$$

Concerning A_4, from the Cauchy-Schwarz inequality and the interpolation error estimates of the linear operator (3.4), it can be estimated as

$$|A_4| \quad = \quad \Big|\big(k(\phi-I_h\phi),p-p_h\big)_{\Gamma_a}\Big| \le Ch^{\gamma_2}\|p-p_h\|_1\|p(q_h)-p_h\|_{0,\Gamma_i}. \tag{4.20}$$

A combination of (4.14), and (4.17)-(4.20), we arrive at the desired error bound for $p(q_h)-p_h$ on the inaccessible boundary Γ_i:

$$\|p(q_h)-p_h\|_{0,\Gamma_i} \le Ch^{\frac{\gamma}{2}}\Big(|||u-u_h|||_1+|||p-p_h|||_1+\tfrac{1}{\beta}\|p-p_h\|_{0,\Gamma_i}\Big), \tag{4.21}$$

with $\gamma=\min\{2\gamma_2,\gamma_1\}$. Finally, by the triangle inequality, (4.6), (4.10), (4.21) and a proper Young's inequality, we can derive

$$\begin{aligned}&\frac{1}{2(1+h^{\gamma_1})}\|u-u_h\|_{0,\Gamma_a}^2\\ &\le \|u(q_h)-u\|_{0,\Gamma_a}^2+\frac{1}{2\beta}\|p(q_h)-p\|_{0,\Gamma_i}^2+Ch^{\gamma_1}|||u-u_h|||_1^2\\ &\quad+\frac{1}{2\beta}\|p(q_h)-p_h\|_{0,\Gamma_i}^2-\frac{1}{4\beta}\|p-p_h\|_{0,\Gamma_i}^2\\ &\le \frac{1}{\beta}\|p(q_h)-p_h\|_{0,\Gamma_i}^2+Ch^{\gamma_1}|||u-u_h|||_1^2-\frac{1}{4\beta}\|p-p_h\|_{0,\Gamma_i}^2\\ &\le Ch^{\gamma}\Big(|||u-u_h|||_1^2+|||p-p_h|||_1^2\Big).\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}&\frac{1}{4}\|p-p_h\|_{0,\Gamma_i}^2\\ &\le \|p-p(q_h)\|_{0,\Gamma_i}^2+\|p(q_h)-p_h\|_{0,\Gamma_i}^2-\frac{1}{4}\|p-p_h\|_{0,\Gamma_i}^2\\ &\le 2\|p(q_h)-p\|_{0,\Gamma_i}^2-\frac{1}{4}\|p-p_h\|_{0,\Gamma_i}^2\\ &\le Ch^{\gamma}\Big(|||u-u_h|||_1^2+|||p-p_h|||_1^2\Big),\end{aligned} \tag{4.23}$$

which imply the desired results of this lemma. Thus the proof is completed. □

Theorem 4.2. *Let (u,p,q) and (u_h,p_h,q_h) be the solutions of (2.3) and (3.3), respectively. Then there exists a constant $C>0$ depending only on the minimum angle of the mesh such that*

$$|||u-u_h|||_1^2 + |||p-p_h|||_1^2 \le C\left(\inf_{v_h\in V_h}|||u-v_h|||_1^2 + \inf_{w_h\in V_h}|||p-w_h|||_1^2\right). \tag{4.24}$$

with the constant $\gamma\in(0,1]$ depending on the geometry of the domain.

Proof. Firstly, by the triangle inequality,

$$|||u-u_h|||_1^2 \le 2|||u-u(q_h)|||_1^2 + 2|||u(q_h)-u_h|||_1^2, \tag{4.25}$$

with $u(q_h)$ defined by the variational equation (4.2).

In order to bound the term $|||u-u(q_h)|||_1^2$, subtracting (4.2) from the first equation of (3.3) and setting $v=u-u(q_h)$, we get

$$\begin{aligned} |||u-u(q_h)|||_1^2 &= -(q-q_h, u-u(q_h))_{\Gamma_i} \\ &\le \|q-q_h\|_{0,\Gamma_i}\|u-u(q_h)\|_{0,\Gamma_i} \\ &\le C\|q-q_h\|_{0,\Gamma_i}|||u-u(q_h)|||_1, \end{aligned} \tag{4.26}$$

which, together with Lemma 4.1, implies that

$$\begin{aligned} |||u-u(q_h)|||_1^2 &\le C\|q-q_h\|_{0,\Gamma_i}^2 \\ &\le Ch^\gamma(|||u-u_h|||_1^2 + |||p-p_h|||_1^2). \end{aligned} \tag{4.27}$$

Since u_h can be viewed as the direct finite element solution of the single elliptic equation (4.2), it follows from the well known error estimate

$$\begin{aligned} |||u(q_h)-u_h|||_1^2 &\le C\inf_{v_h\in V_h}|||u(q_h)-v_h|||_1^2 \\ &\le C\left(\inf_{v_h\in V_h}|||u-v_h|||_1^2 + |||u(q_h)-u|||_1^2\right) \\ &\le C\inf_{v_h\in V_h}|||u-v_h|||_1^2 + Ch^\gamma(|||u-u_h|||_1^2 + |||p-p_h|||_1^2). \end{aligned} \tag{4.28}$$

We next consider $|||p-p_h|||_1^2$. For this purpose, we define $p(u_h)\in H^1(\Omega)$ as the solution of

$$\big(\alpha\nabla p(u_h),\nabla v\big) + \big(kp(u_h),v\big)_{\Gamma_a} = \big(u_h-z,v\big)_{\Gamma_a}, \quad \forall v\in H^1(\Omega). \tag{4.29}$$

The triangle inequality still gives

$$|||p-p_h|||_1^2 \le 2|||p-p(q_h)|||_1^2 + 4|||p(q_h)-p(u_h)|||_1^2 + 4|||p(u_h)-p_h|||_1^2. \tag{4.30}$$

Substituting v with $p-p(q_h)$ in (4.8), we obtain

$$\begin{aligned} |||p-p(q_h)|||_1^2 &= (u-u(q_h),p-p(q_h))_{\Gamma_a} \\ &\le \|u-u(q_h)\|_{0,\Gamma_a}\|p-p(q_h)\|_{0,\Gamma_a} \\ &\le C|||u-u(q_h)|||_1|||p-p(q_h)|||_1, \end{aligned} \tag{4.31}$$

which, together with (4.27), yields

$$\begin{aligned} |||p-p(q_h)|||_1^2 &\le C|||u-u(q_h)|||_1^2 \\ &\le Ch^\gamma(|||u-u_h|||_1^2 + |||p-p_h|||_1^2). \end{aligned} \tag{4.32}$$

Subtracting (4.29) from (4.3) with $v = p(q_h) - p(u_h)$, we have

$$\begin{aligned} |||p(q_h) - p(u_h)|||_1^2 &= (u(q_h) - u_h, p(q_h) - p(u_h))_{\Gamma_a} \\ &\leq \|u(q_h) - u_h\|_{0,\Gamma_a} \|p(q_h) - p(u_h)\|_{0,\Gamma_a} \\ &\leq C|||u(q_h) - u_h|||_1 |||p(q_h) - p(u_h)|||_1, \end{aligned} \tag{4.33}$$

which combined with (4.28), yields

$$\begin{aligned} |||p(q_h) - p(u_h)|||_1^2 &\leq C|||u(q_h) - u_h|||_1^2 \\ &\leq C \inf_{v_h \in V_h} |||u - v_h|||_1^2 + Ch^\gamma (|||u - u_h|||_1^2 + |||p - p_h|||_1^2). \end{aligned} \tag{4.34}$$

Concerned the last term $|||p(u_h) - p_h|||_1^2$, noticing that p_h can be regarded as the finite element approximation of the single equation of (4.29), then there holds

$$\begin{aligned} & |||p(u_h) - p_h|||_1^2 \\ \leq\ & C \inf_{w_h \in V_h} ||||p(u_h) - w_h|||_1^2 \\ \leq\ & C \left(\inf_{w_h \in V_h} |||p - w_h|||_1^2 + |||p(u_h) - p|||_1^2 \right) \\ \leq\ & C \inf_{w_h \in V_h} |||p - w_h|||_1^2 + C(|||p - p(q_h)|||_1^2 + |||p(q_h) - p(u_h)|||_1^2) \\ \leq\ & C \left(\inf_{w_h \in V_h} |||p - w_h|||_1^2 + \inf_{v_h \in V_h} |||u - v_h|||_1^2 \right) + Ch^\gamma (|||u - u_h|||_1^2 + |||p - p_h|||_1^2), \end{aligned} \tag{4.35}$$

where we have used (4.32) and (4.34) in the last step. Lastly, a collection of (4.25), (4.27), (4.28), (4.30), (4.32), (4.34) and (4.35) can yield the desired result (4.24). Hence the proof is completed. □

A straightforward consequence of Theorem 4.2 is the following a priori energy error estimate.

Theorem 4.3. *Let (u, p, q) and (u_h, p_h, q_h) be the solutions of (2.3) and (3.3), respectively. Suppose that $u \in H^{1+\delta}(\Omega)$ and $p \in H^{1+\delta}(\Omega)$ with the constant $\delta \in (0, 1]$. Then there exists a constant $C > 0$ depending only on the minimum angle of the mesh such that*

$$|||u - u_h|||_1^2 + |||p - p_h|||_1^2 \leq Ch^{2\delta} \Big(|u|_{1+\delta}^2 + |p|_{1+\delta}^2 \Big). \tag{4.36}$$

Proof. The proof is trivial and we only need to use the well-known approximation error estimate of the linear finite element space, see, e.g., [6]. □

5. A Priori L^2 Norm Error Estimates

In this section, we derive the L^2 error estimates of the quantity $q - q_h$ on the boundary Γ_i and on the domain Ω, respectively, which are of more interest from the viewpoint of practice.

The following theorem concerning the L^2 error estimate of the quantity $q - q_h$ on the boundary Γ_i is a straightforward corollary of Lemma 4.1 and Theorem 4.3.

Theorem 5.1. *Let (u, p, q) and (u_h, p_h, q_h) be the solutions of (2.3) and (3.3), respectively. Suppose that $u \in H^{1+\delta}(\Omega)$ and $p \in H^{1+\delta}(\Omega)$ with the constant $\delta \in (0, 1]$. Then there exists a constant $C > 0$ depending only on the minimum angle of the mesh such that*

$$\|u - u_h\|_{0,\Gamma_a}^2 + \|p - p_h\|_{0,\Gamma_i}^2 + \|q - q_h\|_{0,\Gamma_i}^2 \leq Ch^{\gamma+2\delta} \Big(|u|_{1+\delta}^2 + |p|_{1+\delta}^2 \Big), \tag{5.1}$$

with the constant $\gamma \in (0, 1]$ depending on the geometry of the domain.

Proof. A combination of (4.1) in Lemma 4.1 and (4.36) in Theorem 4.3 gives us

$$\|u-u_h\|_{0,\Gamma_a}^2+\|p-p_h\|_{0,\Gamma_i}^2 \le Ch^{\gamma+2\delta}\Big(|u|_{1+\delta}^2+|p|_{1+\delta}^2\Big), \tag{5.2}$$

and this implies (5.1) by noticing that $q|_{\Gamma_i}=p|_{\Gamma_i}/\beta$ and $q_h|_{\Gamma_i}=p_h|_{\Gamma_i}/\beta$. □

The following theorem concerning on the L^2-norm error estimate in the entire domain.

Theorem 5.2. *Let (u,p,q) and (u_h,p_h,q_h) be the solutions of (2.3) and (3.3), respectively. Suppose that $u\in H^{1+\delta}(\Omega)$ and $p\in H^{1+\delta}(\Omega)$ with the constant $\delta\in(0,1]$. Then there exists a constant $C>0$ depending only on the minimum angle of the mesh such that*

$$\|u-u_h\|_0^2+\|p-p_h\|_0^2 \quad\le\quad Ch^{\min\{\gamma,2\gamma_3\}+2\delta}\Big(|u|_{1+\delta}^2+|p|_{1+\delta}^2\Big), \tag{5.3}$$

with the constants γ, $\gamma_3\in(0,1]$ depending on the geometry of the domain.

Proof. We consider the following auxiliary problem: Seek $\psi\in H^1(\Omega)$ such that

$$\begin{cases} -\nabla\cdot(\alpha\nabla\psi)=u-u_h, & \text{in } \Omega,\\ \alpha\dfrac{\partial\psi}{\partial n}+k\psi=0, & \text{on } \Gamma_a,\\ \dfrac{\partial\psi}{\partial n}=0, & \text{on } \Gamma_i. \end{cases} \tag{5.4}$$

From [6] we know that there exists a constant $\gamma_3\in(0,1]$ depends on the geometry of the domain such that the above problem have the following regularity result

$$\|\psi\|_{1+\gamma_3}\le C\|u-u_h\|_0\,. \tag{5.5}$$

The variational problem of (5.4) reads as

$$(\alpha\nabla\psi,\nabla v)+(k\psi,v)_{\Gamma_a}=(u-u_h,v),\qquad \forall v\in H^1(\Omega). \tag{5.6}$$

Putting $v=u-u_h$ in (5.6), we have from the first equation in (2.3) and (3.3),

$$\begin{aligned}
&\|u-u_h\|_0^2\\
&=\big(\alpha\nabla\psi,\nabla(u-u_h)\big)+\big(k\psi,u-u_h\big)_{\Gamma_a}\\
&=\big(\alpha\nabla(\psi-I_h\psi),\nabla(u-u_h)\big)+\big(k(\psi-I_h\psi),u-u_h\big)_{\Gamma_a}\\
&\qquad+\big(\alpha\nabla I_h\psi,\nabla(u-u_h)\big)+\big(kI_h\psi,u-u_h\big)_{\Gamma_a}\\
&=\big(\alpha\nabla(\psi-I_h\psi),\nabla(u-u_h)\big)+\big(k(\psi-I_h\psi),u-u_h\big)_{\Gamma_a}-(q-q_h,I_h\psi)_{\Gamma_i}\\
&=\big(\alpha\nabla(\psi-I_h\psi),\nabla(u-u_h)\big)+\big(k(\psi-I_h\psi),u-u_h\big)_{\Gamma_a}\\
&\qquad-(q-q_h,I_h\psi-\psi)_{\Gamma_i}-(q-q_h,\psi)_{\Gamma_i}\\
&\le C\|\psi\|_{1+\gamma_3}\Big(h^{\gamma_3}|||u-u_h|||_1+\|q-q_h\|_{0,\Gamma_i}\Big)\\
&\le C\|u-u_h\|_0\Big(h^{\gamma_3}|||u-u_h|||_1+\|q-q_h\|_{0,\Gamma_i}\Big),
\end{aligned} \tag{5.7}$$

which, together with (4.36) and (5.1), yields

$$\begin{aligned}
\|u-u_h\|_0^2&\le C\Big(h^{2\gamma_3}|||u-u_h|||_1^2+\|q-q_h\|_{0,\Gamma_i}^2\Big)\\
&\le Ch^{\min\{\gamma,2\gamma_3\}+2\delta}\Big(|u|_{1+\delta}^2+|p|_{1+\delta}^2\Big).
\end{aligned} \tag{5.8}$$

Then we only need to estimate the term $\|p-p_h\|_0^2$ in the left. Similarly, we consider the auxiliary problem: Seek $\varphi \in H^1(\Omega)$ such that

$$\begin{cases} -\nabla \cdot (\alpha \nabla \varphi) = p - p_h, & \text{in } \Omega, \\ \alpha \dfrac{\partial \varphi}{\partial n} + k\varphi = 0, & \text{on } \Gamma_a, \\ \dfrac{\partial \varphi}{\partial n} = 0, & \text{on } \Gamma_i, \end{cases} \tag{5.9}$$

with its variational form

$$(\alpha \nabla \varphi, \nabla v) + (k\varphi, v)_{\Gamma_a} = (p - p_h, v), \quad \forall v \in H^1(\Omega), \tag{5.10}$$

and a priori regularity result

$$\|\varphi\|_{1+\gamma_3} \leq C\| |p - p_h\|_0, \tag{5.11}$$

with a constants $\gamma_3 \in (0, 1]$ depending only on the geometry of the domain. Setting $v = p - p_h$ in (5.10), by the second equation in (2.3) and (3.3), we have

$$\begin{aligned} &\|p - p_h\|_0^2 \\ =& \big(\alpha \nabla \varphi, \nabla (p - p_h)\big) + \big(k\varphi, p - p_h\big)_{\Gamma_a} \\ =& \big(\alpha \nabla (\varphi - I_h \varphi), \nabla (p - p_h)\big) + \big(k(\varphi - I_h \varphi), p - p_h\big)_{\Gamma_a} \\ &+ \big(\alpha \nabla I_h \varphi, \nabla (p - p_h)\big) + \big(k I_h \varphi, p - p_h\big)_{\Gamma_a} \\ =& \big(\alpha \nabla (\varphi - I_h \varphi), \nabla (p - p_h)\big) + \big(k(\varphi - I_h \varphi), p - p_h\big)_{\Gamma_a} \\ &+ (u - u_h, I_h \varphi - \varphi)_{\Gamma_a} + (u - u_h, \varphi)_{\Gamma_a} \\ \leq& C\|\varphi\|_{1+\gamma_3} \Big(h^{\gamma_3} \| |p - p_h| \|_1 + \|u - u_h\|_{0,\Gamma_a} \Big) \\ \leq& C\|p - p_h\|_0 \Big(h^{\gamma_3} \| |p - p_h| \|_1 + \|u - u_h\|_{0,\Gamma_a} \Big), \end{aligned} \tag{5.12}$$

which, together with Lemma 4.1 and Theorem 4.3, implies that

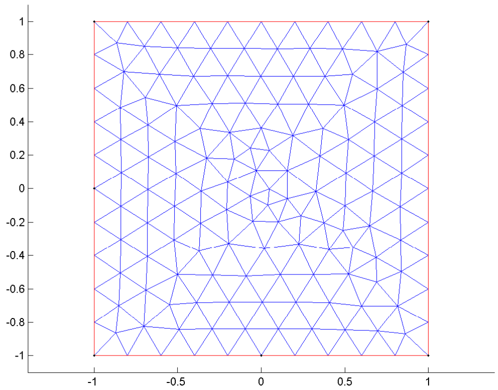
$$\|p - p_h\|_0^2 \leq C h^{\min\{\gamma, 2\gamma_3\} + 2\delta} \Big(|u|_{1+\delta}^2 + |p|_{1+\delta}^2 \Big). \tag{5.13}$$

Therefore the proof is completed. □

Remark 5.3. It is remarked that all the theoretical error decay rates derived in the current and previous sections are the lower bounds under the least regularity assumption of the PDE solutions, which could be improved along with higher regularities of the true solutions of the PDE system (2.3) under concerned. It is therefore not a surprise that one can obtain significantly better convergence rates than predicted on convex domains and smooth data, which is still consistent with our theory.

6. Numerics Tests and Discussions

In this section, we present some numerical examples to verify the decay rates of our theoretical prediction of the error estimates. We will consider two typical domains, namely a square $(-1, 1)^2$ and a L-shaped region $(-1, 1)^2 \setminus (-1, 0)^2$, which are triangulated using an unstructured meshes as shown in Fig. 6.1. Five uniform refinements are done to investigate the decay rates of the finite element approximation error.



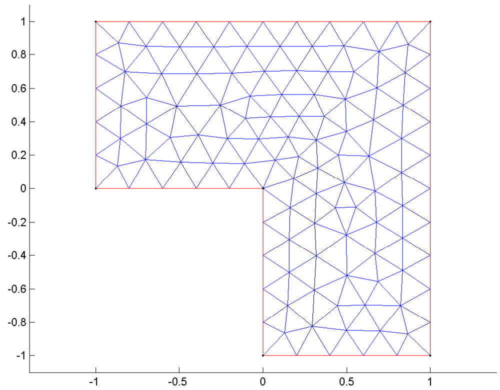


Fig. 6.1. Example 1. Unstructured meshes on computational domains. Left: square; Right: L-shaped region.

In the following, we fix the parameters as follows: the heat conductivity $\alpha = 1$, the heat source $f = 0$, the robin coefficient $k = 1$, the ambient temperature $u_a = 0$, the regularization parameter $\beta = 10^{-2}$ and the noise level $\delta = 10^{-2}$. It is remarked that once the parameters are fixed, the error decay rates depend only on the geometry of domain and given fluxes. For any given flux on Γ_i, specified data aforementioned and the computational domain, it is in general impossible to obtain the explicit formulae for the true solutions u and p. Therefore, we refine the mesh up to 3,400,000 DOF's to yield the approximate *true* solutions, which are compared with those finite element solutions on coarser meshes to investigate the error decay rates. Since the energy norm is equivalent to the H^1-norm. The latter is used instead for convergence tests in the sequel.

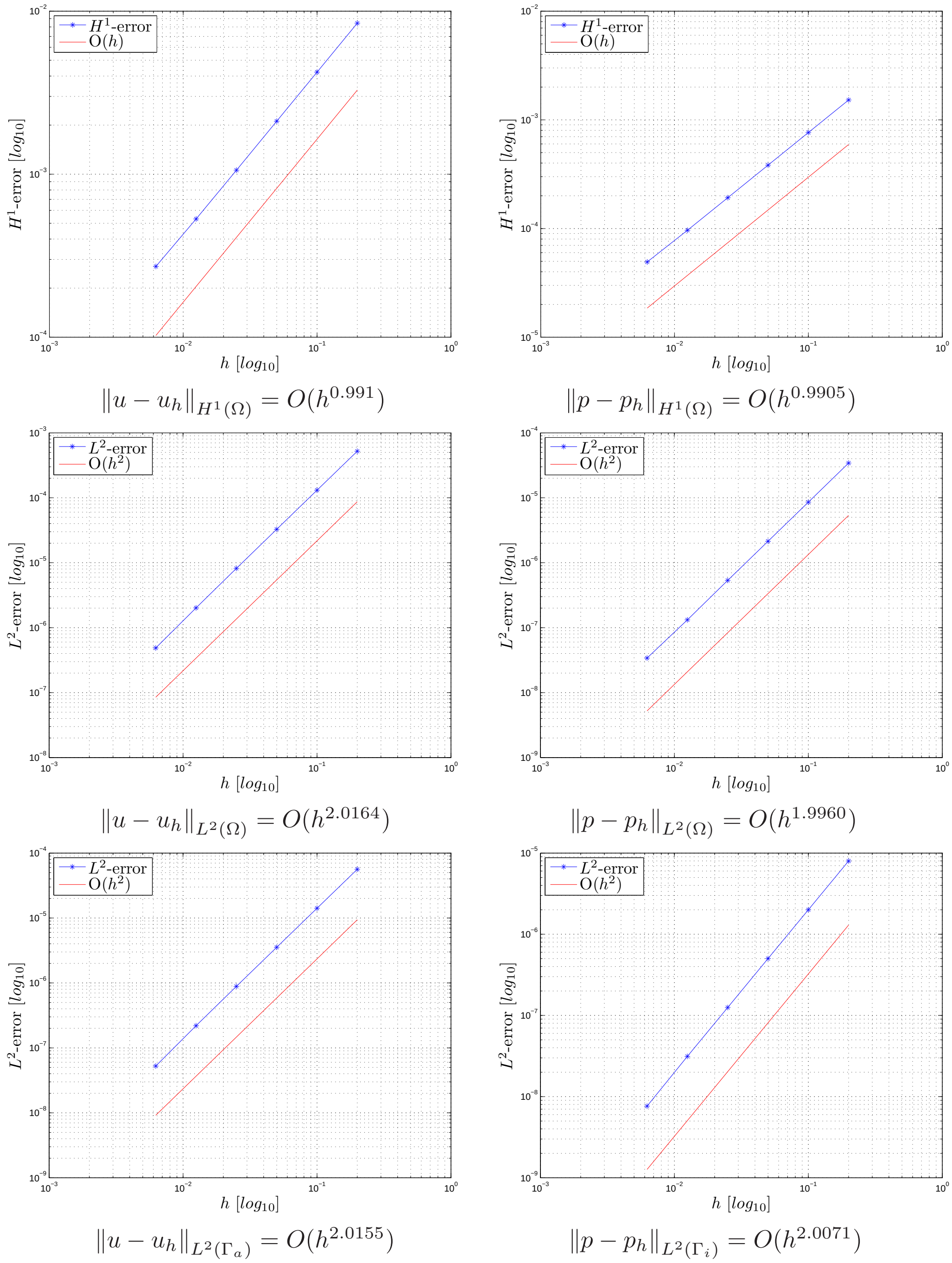
In the tests, the numerical decay rates are obtained by linear regression of the last four groups of data, which are shown right below the convergence history plots and the red lines with specified decay rates are plotted for purpose of reference.

Example 1. Square domain.

In this example, the observation part is chosen to be the bottom boundary $(-1, 1) \times \{-1\}$, and the unknown flux lies on the top boundary $(-1, 1) \times \{1\}$, while the left and right boundaries $\{-1, 1\} \times (-1, 1)$ are set to be heat isulated, namely homogeneous Neumann boundary conditions are used there. We choose a smooth flux data given by the explicit formula $q_e = sin(\pi x)$. It is pointed out that the true flux identified from the system (2.3) is different from q_e due to the regularization effect. The convergence tests are compared with a sufficiently fine finite element solution instead of this exact one.

Since this domain is a convex polygonal, together with smooth flux data, full regularity of the true solutions u and p to the system (2.3) are obtained. We observe in Fig. 6.2 that the decay rates of finite element approximation errors for both u and p are, respectively, close to 1 and 2 in H^1- and L^2-norms. Moreover, we show the decay rates of finite element approximation errors for u and p, respectively, on the measurement boundary Γ_a and interior inaccessible boundary Γ_i in the last row of Fig. 6.2, from which we see clearly second order convergence rates.

On the other hand, we observe that second order of the L^2-norm error decay rates on the inaccessible boundary Γ_i and in the domain, respectively, are optimistic and better than our theoretical prediction in Theorems 5.1 and 5.2. More precisely, the numerical rates outperform the theoretical lower bounds of the convergence order, which is mainly due to the fine property



$\|u-u_h\|_{H^1(\Omega)} = O(h^{0.991})$ $\|p-p_h\|_{H^1(\Omega)} = O(h^{0.9905})$

$\|u-u_h\|_{L^2(\Omega)} = O(h^{2.0164})$ $\|p-p_h\|_{L^2(\Omega)} = O(h^{1.9960})$

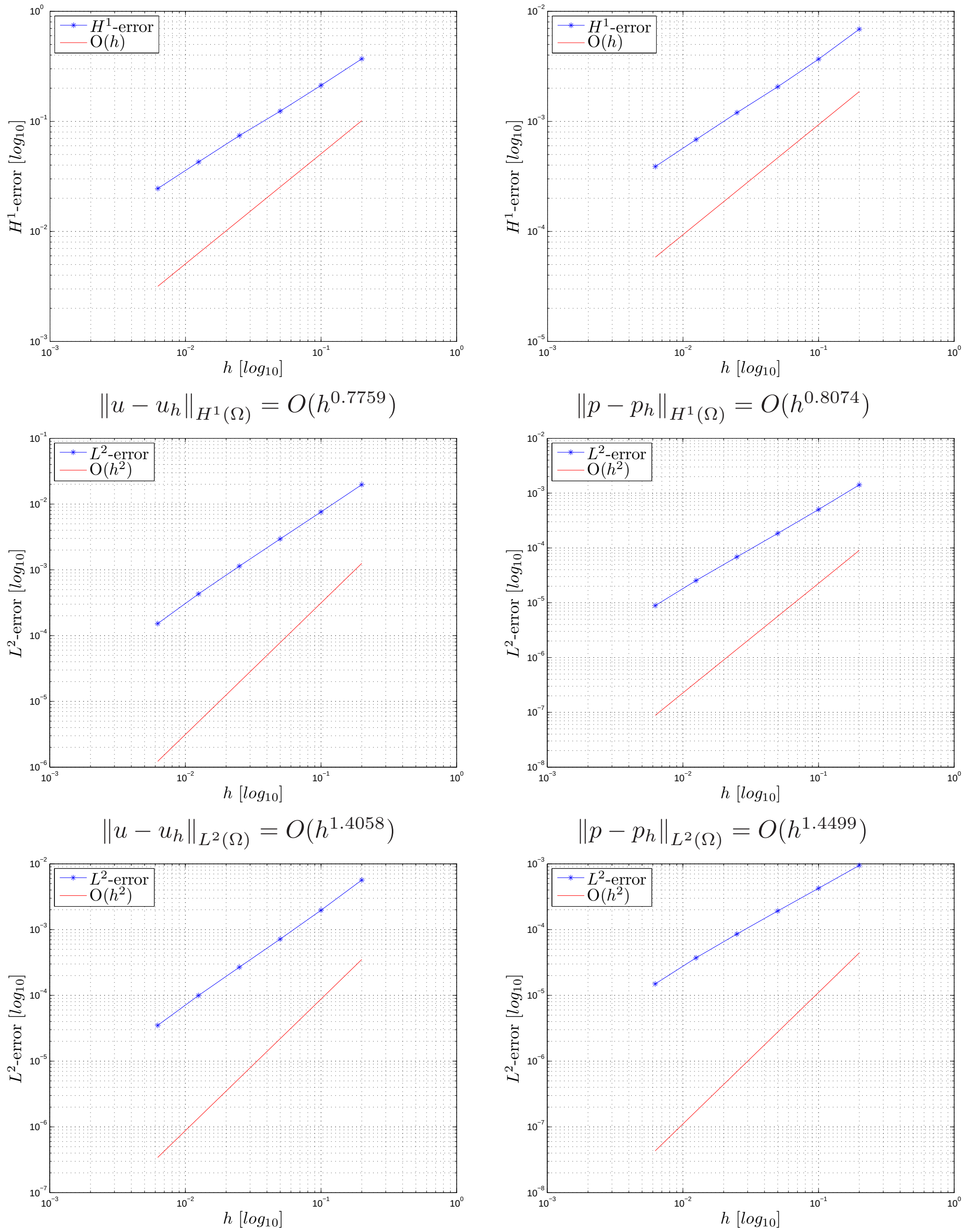
$\|u-u_h\|_{L^2(\Gamma_a)} = O(h^{2.0155})$ $\|p-p_h\|_{L^2(\Gamma_i)} = O(h^{2.0071})$

Fig. 6.2. Example 1. Convergence tests for u and p.

of the geometry of domain and smooth data.

Example 2. L-shaped domain.

The computational domain is non-convex. We choose the observation part to be the top and right outer boundaries $\{1\} \times (-1,1) \cup (-1,1) \times \{1\}$, and the unknown flux is to be determined on the inner boundaries $\{0\} \times (-1,0) \cup (-1,0) \times \{0\}$, while the other two boundaries $\{-1\} \times$



$\|u - u_h\|_{H^1(\Omega)} = O(h^{0.7759})$ $\|p - p_h\|_{H^1(\Omega)} = O(h^{0.8074})$

$\|u - u_h\|_{L^2(\Omega)} = O(h^{1.4058})$ $\|p - p_h\|_{L^2(\Omega)} = O(h^{1.4499})$

$\|u - u_h\|_{L^2(\Gamma_a)} = O(h^{1.4474})$ $\|p - p_h\|_{L^2(\Gamma_i)} = O(h^{1.2024})$

Fig. 6.3. Example 2. Convergence tests for u and p.

$(0,1) \cup (0,1) \times \{-1\}$ are set to be heat isulated, namely homogeneous Neumann boundary conditions are used there. Since this domain is a non-convex region. The exact flux data is given by the explicit formula $q_e = (1-x)^3 + y^2$. We observe in Fig. 6.3 that the decay rates of finite element approximation errors for both u and p in H^1- and L^2-norms, respectively, are significantly reduced from the best case and diverges from the reference red lines. Moreover, we show the decay rates of finite element approximation errors for u and p, respectively, on the measurement boundary Γ_a and interior boundary Γ_i in the last row of Fig. 6.3, from which we see clearly second order convergence rates are no longer obtained for the system on such a L-shaped region.

The decay rate for the L^2-norm error of the adjoint state p demonstrates clearly around order of about 1.2, which is mainly due to the non-convexity of the computational domain. While the L^2-norm error decay rate of u on Γ_i is is also affected due to the geometrical issue and thus reduced to about 1.45.

7. Conclusion

In this work, we established some a priori error estimates for the heat flux reconstruction problem using the continuous linear finite element discretization. These a priori estimates play a key role in proving the convergence analysis of adaptive finite element methods for a general class of inverse heat conduction problems, see, e.g. [17] for the heat flux reconstruction problem. The lower bounds of the convergence order in the L^2-norm are obtained for both u and p in the domain, which is sufficient for an elegant convergence proof of AFEM. Numerical examples are presented to verify our theoretical prediction. Further work on the convergence analysis of adaptive techniques applied to other inverse heat conduction problems will be reported elsewhere.

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