HIGHLY OSCILLATORY DIFFUSION-TYPE EQUATIONS*

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Abstract

We explore new asymptotic-numeric solvers for partial differential equations with highly oscillatory forcing terms. Such methods represent the solution as an asymptotic series, whose terms can be evaluated by solving non-oscillatory problems and they guarantee high accuracy at a low computational cost. We consider two forms of oscillatory forcing terms, namely when the oscillation is in time or in space: each lends itself to different treatment. Numerical examples highlight the salient features of the new approach.

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1. Introduction

Partial differential equations with highly oscillatory forcing terms arise in various branches of science and engineering. In particular, in modern communication systems, high frequencies and signals of widely-varying frequency content abound and present a serious challenge to existing numerical solvers [28, 29]. This is because the highly oscillatory behaviour of the problem compels the use of an exceedingly small step size with such methods. This results in significant accumulation of error and a prohibitive computational workload. The purpose of this paper is to address this issue and develop a numerical methodology which allows for an exceedingly accurate, yet affordable, discretization of partial differential equations with highly oscillatory forcing terms.

The numerical approach presented in this paper is based on the combination of asymptotic and numerical techniques. It involves asymptotic expansions in inverse powers of the oscillatory parameter, ω and numerical discretization of non-oscillatory partial differential equations which

are obtained in the course of the asymptotic expansion. In some situations, this numerical effort can be further reduced by using analytic results.

The core technique has been employed for the numerical approximation of ordinary differential equations that contain highly-oscillatory forcing terms in [12–15]. For example, the solution of the ordinary differential system

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) + \sum_{n=-\infty}^{\infty} \mathbf{b}_n(t)\mathrm{e}^{\mathrm{i}n\omega t}, \quad t \geq 0, \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{1.1}$$

can be expanded asymptotically in the form

$$\mathbf{y}(t) = \mathbf{p}_{0,0}(t) + \sum_{r=1}^{\infty} \frac{1}{\omega^r} \sum_{n=-\infty}^{\infty} \mathbf{p}_{r,n}(t)\mathrm{e}^{\mathrm{i}n\omega t}, \quad t \geq 0, \tag{1.2}$$

where the functions $\mathbf{p}_{r,n}$ are independent of ω, hence non-oscillatory, and can be obtained for $n = 0$ by solving a non-oscillatory ordinary differential equation and by recursion for $n \neq 0$ [15].

The asymptotic expansion (1.2) has been shown to have very important benefits compared to standard discretization methods for ordinary differential equations. Firstly, the method is considerably more efficient for large values of the oscillatory parameter. Secondly, the computational effort is independent of the value of that parameter [12]. This success motivates our effort to extend it from systems of the form (1.1) to partial differential equations, the subject matter of this paper.

The current paper concentrates on the diffusion equation with Dirichlet or Neumann boundary conditions in the interval $[-1, 1]$ and it is concerned with two types of oscillatory terms, namely when the oscillation occurs in time or in space. We demonstrate that a model similar to (1.2) falls short of describing a solution of a diffusion equation with a highly-oscillatory forcing term but a considerably more complicated *ansatz* is equal to this task. We also observe an interesting phenomenon: while in the case of time-like oscillations in the forcing term, the solution, similar to (1.2), consists of a non-oscillatory component overlaid with small amplitude oscillations, the solution for a forcing term oscillating in space is itself non-oscillatory!

The extension of the methodology underlying (1.2), blending asymptotic and numerical techniques, into the realm of partial differential equations is far from simple. In particular, it necessitates the solution of initial-boundary value problems which, although non-oscillatory, present us with a major challenge once we wish to obtain high accuracy. This is further elaborated in the sequel.

Interesting applications of partial differential equations with highly oscillatory forcing terms are not restricted to the diffusion equation: a case in point is computational electronics, where the differential operator is hyperbolic [29]. We can also remark that Modified Fourier Expansions have been developed to treat analytical and numerical problems in partial differential equations [6–11]. However, this type of expansion uses a different kind of approach and it has been used in a different kind of problem setting to the one we address in this paper. In the current paper we consider a special case, while endeavouring to establish a general framework relevant to other highly oscillatory equations.

The theory of Laplace–Dirichlet and Laplace–Neumann expansions in parallelepipeds is quite comprehensively understood (e.g., [1]). We recognise that the extension of the expansions in this paper to several dimensions is likely to result in fairly unpleasant expressions, but this is a technical, rather than conceptual difficulty. We note that such expansions have been also

extended by Huybrechs et al. [16] to triangles: we do not mention this in the current paper because this would be an unnecessary distraction. It is true that our expansions are unlikely to be practical in general domains or in a free-boundary setting. Having said so, as we have already mentioned, this is a preliminary paper, addressing such problems for the first time in an organised manner, and one has to start somewhere.

2. Time-Like Oscillations

2.1. The general framework

We consider the univariate diffusion equation with a time-like oscillatory forcing term,

$$\partial_t u = \partial_x^2 u + \sum_{n=-\infty}^{\infty} b_n(x,t)e^{in\omega t}, \quad x \in [-1,1], \quad t \geq 0. \tag{2.1}$$

given with the initial condition

$$u(x,0) = \phi(x), \qquad x \in [-1,1], \tag{2.2}$$

and either Dirichlet,

$$u(\pm 1, t) = \nu_{\pm}(t), \qquad t \geq 0, \tag{2.3}$$

or Neumann

$$\partial_x u(\pm 1, t) = \mu_{\pm}(t), \qquad t \geq 0, \tag{2.4}$$

boundary conditions. We assume for simplicity that the functions b_n, ϕ, $\nu_{\pm}$ and $\mu_{\pm}$ are all analytic but our results can be extended to sufficiently differentiable functions in a transparent manner. Moreover, we assume that the functions b_n decay sufficiently rapidly for $|n| \gg 1$, rendering the infinite sum convergent.

Commencing from Dirichlet boundary conditions (2.3), we seek a solution of the form

$$u(x,t) = p_0(x,t) + \sum_{n \neq 0} p_n(x,t,\omega)e^{in\omega t}, \quad t \geq 0, \quad x \in [-1,1], \tag{2.5}$$

where p_0 and u share initial and boundary conditions,

$$p_0(x,0) = \phi(x), \quad x \in [-1,1], \qquad p_0(\pm 1, t) = \nu_{\pm}(t), \quad t \geq 0,$$

while each p_n, $n \neq 0$, obeys zero initial and boundary conditions. Note that the p_ns for $n \neq 0$ are allowed to depend upon ω, indeed be oscillatory. Substituting (2.5) into (2.1) results in

$$\partial_t p_0 + \omega \sum_{n \neq 0} in p_n e^{in\omega t} + \sum_{n \neq 0} (\partial_t p_n)e^{in\omega t}$$

$$= \partial_x^2 p_0 + \sum_{n \neq 0} (\partial_x^2 p_n)e^{in\omega t} + \sum_{n=-\infty}^{\infty} b_n e^{in\omega t}.$$

Thus, separating frequencies, we obtain the following equations,

$$\partial_t p_0 = \partial_x^2 p_0 + b_0, \qquad t \geq 0, \qquad x \in [-1,1], \tag{2.6}$$

where

$$p_0(x,0) = \phi(x), \quad x \in [-1,1], \qquad p_0(\pm 1, t) = \nu_\pm(t), \quad t \ge 0,$$
$$\partial_t p_n = (\partial_x^2 - \mathrm{i} n \omega) p_n + b_n, \qquad t \ge 0, \quad x \in [-1,1], \tag{2.7}$$

where

$$p_n(x,0) \equiv 0, \quad x \in [-1,1], \qquad p_n(\pm 1, t) \equiv 0, \quad t \ge 0,$$

for every $n \ne 0$.

2.2. The computation of p_0

There are several alternative ways of computing (2.6), but in this paper we seek to derive a solution which also makes sense as an analytic construct, so that we can make better sense of the properties of the functions p_n, $n \in \mathbb{Z}$. It is convenient to this end to seek an explicit expansion of the solution. A prime contender is a spectral method, expanding p_0, say, as a linear combination of Chebyshev polynomials. However, this leads to unwieldy expansion terms, which cannot be written explicitly, as well as to possible instability once the expansion is truncated [4,17,19]. We employ instead *Birkhoff expansions*, i.e. expansions in an orthonormal basis of eigenfunctions of a differential operator [27]. Specifically, we use the eigenvalues of the differential operator ∂_x^2 in $[-1,1]$, accompanied by zero Dirichlet boundary conditions: the *Laplace-Dirichlet basis* [1]. Given a function $f \in L_2[-1,1]$ which vanishes at ± 1, we expand

$$f(x) \sim \sum_{m=0}^\infty f_m^C \cos \pi (m + \tfrac{1}{2}) x + \sum_{m=1}^\infty f_m^S \sin \pi m x, \tag{2.8}$$

where

$$f_m^C = \int_{-1}^1 f(x) \cos \pi (m + \tfrac{1}{2}) x \mathrm{d}x, \quad m \in \mathbb{Z}_+,$$
$$f_m^S = \int_{-1}^1 f(x) \sin \pi m x \mathrm{d}x, \quad m \in \mathbb{N}.$$

The expansion (2.8) converges for all $f \in \mathbb{C}^1[-1,1]$ [1]. The expansion coefficients can be computed rapidly with the Fast Fourier Transform, alternatively (and even faster) they can be approximated using asymptotic expansions of highly oscillatory integrals, similarly to the Laplace-Neumann expansions in [1,25].

However, before we expand the solution of (2.1), we need to induce zero Dirichlet boundary conditions at the price of amending the forcing term. Thus, let

$$\tilde{p}_0(x,t) = p_0(x,t) - \frac{1}{2}(1-x)\nu_-(t) - \frac{1}{2}(1+x)\nu_+(t), \quad t \ge 0, \quad x \in [-1,1].$$

Trivially, it obeys the diffusion-type equation

$$\partial_t \tilde{p}_0 = \partial_x^2 \tilde{p}_0 + \tilde{b}_0, \quad t \ge 0, \quad x \in [-1,1], \tag{2.9}$$

where

$$\tilde{p}_0(x,0) = \tilde{\phi}(x), \quad x \in [-1,1], \qquad \tilde{p}_0(\pm 1, t) \equiv 0, \quad t \ge 0,$$

and

$$\tilde{b}_0(x,t) = b_0(x,t) - \tfrac{1}{2}(1-x)\nu'_-(t) - \tfrac{1}{2}(1+x)\nu'_+(t), \quad t \geq 0, \quad x \in [-1,1],$$

$$\tilde{\phi}(x) = \phi(x) - \frac{1}{2}(1-x)\nu_-(0) - \frac{1}{2}(1+x)\nu_+(0), \qquad x \in [-1,1].$$

Assuming that

$$\tilde{b}_0(x,t) = \sum_{m=0}^{\infty} \alpha_{0,m}(t) \cos \pi(m+\tfrac{1}{2})x + \sum_{m=1}^{\infty} \beta_{0,m}(t) \sin \pi m x,$$

$$\tilde{\phi}_0(x) = \sum_{m=0}^{\infty} c_m \cos \pi(m+\tfrac{1}{2})x + \sum_{m=1}^{\infty} d_m \sin \pi m x,$$

where

$$\alpha_{0,m}(t) = \int_{-1}^{1} \tilde{b}_0(x,t) \cos \pi(m+\tfrac{1}{2})x \mathrm{d}x, \quad \beta_{0,m}(t) = \int_{-1}^{1} \tilde{b}_0(x,t) \sin \pi m x \mathrm{d}x,$$

$$c_m = \int_{-1}^{1} \tilde{\phi}(x) \cos \pi(m+\tfrac{1}{2})x \mathrm{d}x, \qquad d_m = \int_{-1}^{1} \tilde{\phi}(x) \sin \pi m x \mathrm{d}x,$$

we seek the expansion

$$\tilde{p}_0(x,t) = \sum_{m=0}^{\infty} \rho_{0,m} \cos \pi(m+\tfrac{1}{2})x + \sum_{m=1}^{\infty} \sigma_{0,m} \sin \pi m x.$$

Substitution in (2.9) yields the scalar ordinary differential equations whose explicit solution are

$$\rho_{0,m}(t) = \mathrm{e}^{-\pi^2\left(m+\frac{1}{2}\right)^2 t} c_m + \int_0^t \mathrm{e}^{-\pi^2\left(m+\frac{1}{2}\right)^2 (t-\tau)} \alpha_{0,m}(\tau) \mathrm{d}\tau, \quad m \in \mathbb{Z}_+,$$

$$\sigma_{0,m}(t) = \mathrm{e}^{-\pi^2 m^2 t} d_m + \int_0^t \mathrm{e}^{-\pi^2 m^2 (t-\tau)} \beta_{0,m}(\tau) \mathrm{d}\tau, \quad m \in \mathbb{N}.$$

Note that

$$p_0(x,t) = \frac{1}{2}(1-x)\nu_-(t) + \tfrac{1}{2}(1+x)\nu_+(t)$$

$$+ \sum_{m=0}^{\infty} \rho_{0,m}(t) \cos \pi(m+\tfrac{1}{2})x + \sum_{m=1}^{\infty} \sigma_{0,m}(t) \sin \pi m x, \qquad (2.10)$$

is essentially a consequence of the Duhamel principle [26], rendered explicitly and in a form which will be found convenient in the sequel. Note further that, being independent of ω, p_0 does not oscillate.

2.3. The computation of p_n, $n \neq 0$

The equation (2.7) is already equipped with zero boundary conditions, and this makes its Laplace-Dirichlet expansion somewhat more transparent. Thus, expanding

$$b_n(x,t) = \sum_{m=0}^{\infty} \alpha_{n,m}(t) \cos \pi(m+\tfrac{1}{2})x + \sum_{m=1}^{\infty} \beta_{n,m}(t) \sin \pi m x,$$

we seek $\rho_{n,m}(t)$, $m \in \mathbb{Z}_+$, and $\sigma_{n,m}(t)$, $m \in \mathbb{N}$, such that

$$p_n(x,t) = \sum_{m=0}^{\infty} \rho_{n,m}(t) \cos \pi(m + \tfrac{1}{2})x + \sum_{m=1}^{\infty} \sigma_{n,m}(t) \sin \pi m x.$$

Substitution in (2.7) results in ordinary differential equations whose explicit solutions are

$$\rho_{n,m}(t) = \mathrm{e}^{-\left[\pi^2\left(m+\frac{1}{2}\right)^2 + \mathrm{i}n\omega\right]t} \int_0^t \mathrm{e}^{\left[\pi^2(m+\frac{1}{2})^2 + \mathrm{i}n\omega\right]\tau} \alpha_{n,m}(\tau)\mathrm{d}\tau, \quad m \in \mathbb{Z}_+,$$

$$\sigma_{n,m}(t) = \mathrm{e}^{-[\pi^2 m^2 + \mathrm{i}n\omega]t} \int_0^t \mathrm{e}^{[\pi^2 m^2 + \mathrm{i}n\omega]\tau} \beta_{n,m}(\tau)\mathrm{d}\tau, \quad m \in \mathbb{N}.$$

This, however, is not the end of the story, because the $\rho_{n,m}$ and $\sigma_{n,m}$ are expressed above as highly oscillatory integrals. To render the oscillation tractable, we expand them into asymptotic series. It is already known from [24] that

$$\int_0^t g(\tau)\mathrm{e}^{\eta\tau}\mathrm{d}\tau \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{\eta^{k+1}}[g^{(k)}(t)\mathrm{e}^{\eta t} - g^{(k)}(0)], \quad |\eta| \gg 1, \tag{2.11}$$

for any $g \in C^{\infty}[0,t]$.

We first let $\eta = \pi^2(m + \tfrac{1}{2})^2 + \mathrm{i}n\omega$, $g = \alpha_{n,m}$, whereby

$$\rho_{n,m}(t) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{\left[\pi^2\left(m+\frac{1}{2}\right)^2 + \mathrm{i}n\omega\right]^{k+1}} \left\{ \alpha_{n,m}^{(k)}(t) - \alpha_{n,m}^{(k)}(0)\mathrm{e}^{-\left[\pi^2\left(m+\frac{1}{2}\right)^2 + \mathrm{i}n\omega\right]t} \right\}. \tag{2.12}$$

Likewise, $\eta = \pi^2 m^2 + \mathrm{i}n\omega$, $g = \beta_{n,m}$, results in

$$\sigma_{n,m}(t) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{[\pi^2 m^2 + \mathrm{i}n\omega]^{k+1}} \left\{ \beta_{n,m}^{(k)}(t) - \beta_{n,m}^{(k)}(0)\mathrm{e}^{-[\pi^2 m^2 + \mathrm{i}n\omega]t} \right\}. \tag{2.13}$$

2.4. Assembling the solution

Substituting the explicit values of $p'_n s$ from (2.10), (2.12)–(2.13) into (2.5) gives

$$u(x,t) = p_0(x,t) + \sum_{n \neq 0} p_n(x,t)\mathrm{e}^{\mathrm{i}n\omega t}$$

$$\sim \frac{1}{2}(1-x)\nu_-(t) + \frac{1}{2}(1+x)\nu_+(t)$$

$$+ \sum_{m=0}^{\infty} \left\{ \sum_{n \neq 0} \sum_{k=0}^{\infty} \frac{(-1)^k}{[\pi^2(m+\frac{1}{2})^2 + \mathrm{i}n\omega]^{k+1}} \left[\alpha_{n,m}^{(k)}(t)\mathrm{e}^{\mathrm{i}n\omega t} - \alpha_{n,m}^{(k)}(0)\mathrm{e}^{-\pi^2(m+\frac{1}{2})^2 t} \right] \right.$$

$$\left. + \mathrm{e}^{-\pi^2(m+\frac{1}{2})^2 t} c_m + \int_0^t \mathrm{e}^{-\pi^2(m+\frac{1}{2})^2(t-\tau)} \alpha_{0,m}(\tau)\,\mathrm{d}\tau \right\} \cos \pi(m + \tfrac{1}{2})x$$

$$+ \sum_{m=1}^{\infty} \left\{ \sum_{n \neq 0} \sum_{k=0}^{\infty} \frac{(-1)^k}{[\pi^2 m^2 + \mathrm{i}n\omega]^{k+1}} \left[\beta_{n,m}^{(k)}(t)\mathrm{e}^{\mathrm{i}n\omega t} - \beta_{n,m}^{(k)}(0)\mathrm{e}^{-\pi^2 m^2 t} \right] \right.$$

$$\left. + \mathrm{e}^{-\pi^2 m^2 t} d_m + \int_0^t \mathrm{e}^{-\pi^2 m^2(t-\tau)} \beta_{0,m}(\tau)\,\mathrm{d}\tau \right\} \sin \pi m x.$$

We deduce that

$$u(x,t) = q_0(x,t) + \sum_{n \neq 0} q_n(x,t) e^{in\omega t}, \tag{2.14}$$

where

$$q_0(x,t) \sim \frac{1}{2}(1-x)\nu_-(t) + \frac{1}{2}(1+x)\nu_+(t) + \sum_{m=0}^{\infty} e^{-\pi^2(m+\frac{1}{2})^2 t} \Bigg\{ c_m \tag{2.15}$$

$$+ \int_0^t e^{\pi^2(m+\frac{1}{2})^2 \tau} \alpha_{0,m}(\tau)\,\mathrm{d}\tau - \sum_{n\neq0}\sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{n,m}^{(k)}(0)}{[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}} \Bigg\} \cos \pi(m+\tfrac{1}{2})x$$

$$+ \sum_{m=1}^{\infty} e^{-\pi^2 m^2 t} \Bigg\{ d_m + \int_0^t e^{\pi^2 m^2 \tau} \beta_{0,m}(\tau)\,\mathrm{d}\tau - \sum_{n\neq0}\sum_{k=0}^{\infty} \frac{(-1)^k \beta_{n,m}^{(k)}(0)}{[\pi^2 m^2 + in\omega]^{k+1}} \Bigg\} \sin \pi m x,$$

$$q_n(x,t) \sim \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{n,m}^{(k)}(t)}{[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}} \cos \pi(m+\tfrac{1}{2})x$$

$$+ \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k \beta_{n,m}^{(k)}(t)}{[\pi^2 m^2 + in\omega]^{k+1}} \sin \pi m x, \qquad n \neq 0. \tag{2.16}$$

Note that, although the q_ns depend upon ω, they are not oscillatory in ω.

It is possible to expand all terms such that $\pi^2 m^2 < |n|\omega$ asymptotically in inverse powers of ω, while if $\pi^2(m+\frac{1}{2})^2 > |n|\omega$ then terms can be expanded in inverse powers of m. This is a tell-tale indicator of the fact that we have two highly oscillatory phenomena intermingling: the high oscillation originating in ω and the high oscillation of the basis terms $\cos \pi(m+\frac{1}{2})x$ and $\sin \pi m x$. Such expansions needlessly complicate matters and we will not pursue this route further.

The explicit expansions (2.15) and (2.16) indicate that the components of $q_0 - \frac{1}{2}(1-x)\nu_- - \frac{1}{2}(1+x)\nu_+$ corresponding to different values of m decay rapidly for increasing t: this, of course, is a consequence of the rapid dissipation of the diffusion operator. The coefficients q_n, $n \neq 0$, of oscillatory terms, however, need not exhibit decay for increasing t. However, its leading term being independent of ω, q_0 is $\mathcal{O}(1)$ in ω, while q_n for $n \neq 0$ is $\mathcal{O}(\omega^{-1})$, we expect the solution to consist of a non-oscillatory signal of larger amplitude, with rapid, small amplitude oscillations superimposed upon it. This is precisely the situation for ordinary differential equations fitting into model (1.2) and is consistent with Fig. 2.1.

2.5. Implementation issues

We compute $u(x,t)$ in a time-stepping manner for $\{t_N\}_{N\in\mathbb{N}}$, where $t_{N+1} = t_N + \Delta t_N$. An obvious reason is that the numerical calculation of the non-oscillatory integrals in (2.15) by quadrature is best done in fairly small intervals, but the rationale for time-stepping is more subtle. The expansions (2.15) and (2.16) are asymptotic and in general it might be too much to expect that they converge (or give good results upon truncation) for all $t \geq 0$. Therefore it is a good strategy to 'restart' the asymptotic expansion once in a while. Having said so, the steps Δt_ℓ need not be small, since their magnitude does not play a major role in keeping the error at bay.

In what follows we describe the $N = 0$ step, the generalisation to other values of N being straightforward. The construction of (2.14) in a form amenable to numerical computation requires a number of considerations.

Firstly, the range of n needs to be truncated. This is often not a problem because many oscillators of practical interest are of *bounded band-width:* only a finite number of b_ns are nonzero. Otherwise, the size of such truncation depends on the rate of decay of the b_ns as $|n| \to \infty$.

Secondly, we need to truncate the Laplace–Dirichlet expansion. This, of course, depends on the functions whose expansions need to be computed: $\tilde{b}_0$, b_n for $n \neq 0$ and $\tilde{\phi}$. In general, the speed of convergence in (2.8), even for very smooth functions, is slow and, once truncated for $m \leq M$, it might be as slow as $\mathcal{O}(M^{-1})$ even for an analytic function f [1]. Here, perhaps paradoxically, we are saved by high oscillation. Suppose, thus, that we wish to compute (2.14) at $\Delta t_0 > 0$. The mth term in (2.15) is then attenuated by either $\exp(-\pi^2(m + \frac{1}{2})^2 \Delta t_0)$ or $\exp(-\pi^2 m^2 \Delta t_0)$ while the terms in (2.16) are attenuated by either $[\pi^2(m + \frac{1}{2})^2 + in\omega]^{-1}$ or $(\pi^2 m^2 + in\omega)^{-1}$. Thus, for example, we need not have to take M large enough so that, for example, $|\alpha_{n,m}|$ is suitably small: it is enough that $|\alpha_{n,m}/[\pi^2(m + \frac{1}{2})^2 + in\omega]|$ is small.

Another mechanism accelerating the convergence in (2.15) is that $\tilde{\phi}(\pm 1) = 0$. Thus, twice integrating by parts,

$$
\begin{aligned}
c_m &= \frac{1}{\pi(m + \frac{1}{2})} \int_{-1}^{1} \tilde{\phi}(x) \frac{\mathrm{d}\sin\pi(m + \frac{1}{2})x}{\mathrm{d}x} \, \mathrm{d}x \\
&= -\frac{1}{\pi(m + \frac{1}{2})} \int_{-1}^{1} \tilde{\phi}'(x) \sin\pi(m + \tfrac{1}{2})x \, \mathrm{d}x \\
&= \frac{1}{\pi^2(m + \frac{1}{2})^2} \int_{-1}^{1} \tilde{\phi}'(x) \frac{\mathrm{d}\cos\pi(m + \frac{1}{2})x}{\mathrm{d}x} \, \mathrm{d}x \\
&= -\frac{1}{\pi^2(m + \frac{1}{2})^2} \int_{-1}^{1} \phi''(x) \cos\pi(m + \tfrac{1}{2})x \, \mathrm{d}x = \mathcal{O}(m^{-3}),
\end{aligned}
$$

likewise $d_m = \mathcal{O}(m^{-3})$.

Suppose that we truncate $m \leq M$, $|n| \leq N$ and $k \leq K$: note that in our numerical experiments $K = 2$ already ensures good accuracy. We then need to compute

$$
c_m = \int_{-1}^{1} \tilde{\phi}(x) \cos\pi(m + \tfrac{1}{2})x \, \mathrm{d}x, \qquad 0 \leq m \leq M,
$$

$$
d_m = \int_{-1}^{1} \tilde{\phi}(x) \sin\pi m x \, \mathrm{d}x, \qquad\qquad 1 \leq m \leq M,
$$

$$
\alpha_{n,m}^{(k)}(t) = \int_{-1}^{1} \partial_t^k b_n(x, t) \cos\pi(m + \tfrac{1}{2})x \, \mathrm{d}x, \qquad |n| \leq N, \quad 0 \leq m \leq M, \quad 0 \leq k \leq K,
$$

$$
\beta_{n,m}^{(k)}(t) = \int_{-1}^{1} \partial_t^k b_n(x, t) \sin\pi m x \, \mathrm{d}x, \qquad\qquad |n| \leq N, \quad 0 \leq m \leq M, \quad 0 \leq k \leq K,
$$

except that for $n = 0$ we need to replace b_0 with $\tilde{b}_0$. This can be accomplished with $2 + 2(2N + 1)(K + 1)$ FFTs of length M: note that M is likely to be large, while both N and K are typically small. Alternatively, we can use special quadrature methods to compute Birkhoff expansions along the same lines as in [23].

Bearing in mind that $\tilde{\phi} = \phi - \frac{1}{2}(1 - x)\phi(-1) - \frac{1}{2}(1 + x)\phi(1)$ and $\tilde{b}_0 = b_0 + \frac{1}{2}(1 - x)\nu'_- + \frac{1}{2}(1 + x)\nu'_+$, we can remove the tildes from the above formulæ by using the explicit expansions

$$
\frac{1 - x}{2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\pi(m + \frac{1}{2})} \cos\pi(m + \tfrac{1}{2})x + \sum_{m=1}^{\infty} \frac{(-1)^m}{\pi m} \sin\pi m,
$$

$$\frac{1+x}{2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\pi(m+\frac{1}{2})} \cos \pi(m+\tfrac{1}{2})x - \sum_{m=1}^{\infty} \frac{(-1)^m}{\pi m} \sin \pi m.$$

Finally, we need to compute integrals of the form

$$\int_0^t e^{\pi^2(m+\frac{1}{2})^2 \tau} \alpha_{0,m}(\tau)\, d\tau, \qquad \int_0^t e^{\pi^2 m^2 \tau} \beta_{0,m}(\tau)\, d\tau.$$

Given their non-oscillatory nature, they can be computed by any standard quadrature methods, e.g. Gauss–Legendre quadrature in $[0, t]$.

2.6. A worked-out example

Let us consider the equation

$$\partial_t u = \partial_x^2 u + e^{-\pi^2 t} \cos \pi x \cos \omega t, \qquad t \geq 0, \quad t \in [-1, 1], \tag{2.17a}$$

$$u(x, 0) = \cos 2\pi x, \quad x \in [-1, 1], \tag{2.17b}$$

$$u(\pm 1, t) = e^{-\pi^2 t}, \quad t \geq 0. \tag{2.17c}$$

Therefore $b_{\pm 1}(x, t) = \frac{1}{2} e^{-\pi^2 t} \cos \pi x$, $b_n \equiv 0$ for $|n| \neq 1$, $\phi(x) = \cos 2\pi x$, $\tilde{b}_0 = \pi^2 e^{-\pi^2 t}$ and $\nu_{\pm}(t) = e^{-\pi^2 t}$. Now, we need to compute c_m, d_m, $\alpha_{n,m}^{(k)}$ and $\beta_{n,m}^{(k)}$. The details of these calculations are given in Appendix A. Substituting them into (2.15), we have

$$q_0(x, t) = e^{-\pi^2 t} \cos \pi x + 2 \sum_{m=0}^{\infty} (-1)^m e^{-\pi^2(m+\frac{1}{2})^2 t} \left[\frac{4}{\pi(m-\frac{3}{2})(m+\frac{1}{2})(m+\frac{5}{2})} \right.$$

$$\left. - \frac{1}{\pi(m-\frac{1}{2})(m+\frac{1}{2})(m+\frac{3}{2})} - \frac{2\pi(m+\frac{1}{2})e^{-\pi^2 t}}{\pi^4(m-\frac{1}{2})^2(m+\frac{3}{2})^2 + \omega^2} \right] \cos \pi(m+\tfrac{1}{2})x,$$

and

$$\sum_{n \neq 0} q_n(x, t) e^{in\omega t} \sim -2\pi e^{-\pi^2 t} \cos \omega t \sum_{m=0}^{\infty} \frac{(-1)^m (m+\frac{1}{2})}{\pi^4(m-\frac{1}{2})^2(m+\frac{3}{2})^2 + \omega^2} \cos \pi(m+\tfrac{1}{2})x$$

$$- 2\omega e^{-\pi^2 t} \sin \omega t \sum_{m=0}^{\infty} \frac{(-1)^m(m+\frac{1}{2})}{\pi(m-\frac{1}{2})(m+\frac{3}{2})} \cdot \frac{\cos \pi(m+\frac{1}{2})x}{\pi^4(m-\frac{1}{2})^2(m+\frac{3}{2})^2 + \omega^2}.$$

Note that all the expansion terms in the different components of u decay at least as fast as $\mathcal{O}(m^{-3})$ for $m \gg 1$. In particular, individual components in the oscillatory term

$$q^o(x, t) = \sum_{n \neq 0} q_n(x, t) e^{in\omega t}$$

decay like $\mathcal{O}\left(\max\{\min\{m^{-3}, m\omega^{-2}\}, \min\{\omega m^{-5}, m\omega^{-1}\}\}\right)$.

Fig. 2.1 depicts the non-oscillatory and oscillatory components of the solution of (2.17) for $\omega = 10^j$, $j = 1, 2, 3$. Note that we have plotted q_0 for $t \in [0, \frac{1}{2}]$ and q^o just for $t \in [0, \frac{1}{10}]$, to highlight more easily salient features of the two functions.

It is evident that the non-oscillatory component q_0 exhibits very weak dependence on ω, while the oscillatory component q^o changes radically with ω. Moreover, the amplitude of q^o decays like $\mathcal{O}(\omega^{-1})$ – all this is exactly in line with the prediction of our theory and with the
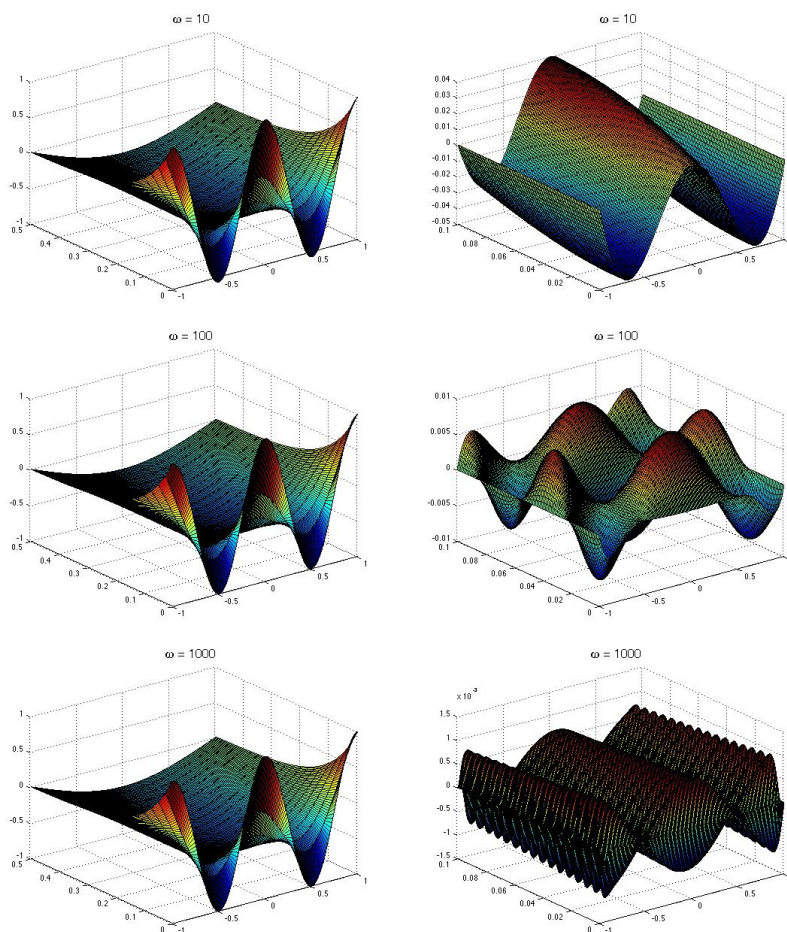
Fig. 2.1. *The non-oscillatory (on the left) and oscillatory components of $u(x, t)$, the solution of (2.17), for different values of ω.*

explicit expressions above. Further note that this amplitude is roughly constant in time, while q_0 tends at an exponential speed to a limit, something which follows from the general theory and upon which we have already commented. This is also evident in Fig. 2.2.

Another observation upon careful perusal of Fig. 2.1 is that the oscillation in q^o (q_0 is of course oscillation-free) takes place in time (for sufficiently large ω), but not in space. This should be apparent from the explicit form of the solution and makes intuitive sense because the forcing term in (2.17) oscillates solely in time.

On the face of it, q^o contributes very little to the solution u, in particular for large $\omega > 0$: after all, typically partial differential equations are solved numerically to fairly coarse tolerance and an error of 10^{-3}, say, for $\omega = 1000$ might well be acceptable. This is a fair point and often it is perfectly reasonable to approximate u by p_0. However, one should not deduce from this that it is a good idea to solve (2.1) by a conventional numerical method, e.g. finite difference, finite element or spectral. Once the space derivative has been discretized, we obtain a highly oscillatory set of ordinary differential equations. Such equations can be solved by classical methods, no matter how 'stable' in a conventional sense, only with a time step of magnitude $\mathcal{O}(\omega^{-1})$. This makes for an exceedingly expensive computation.
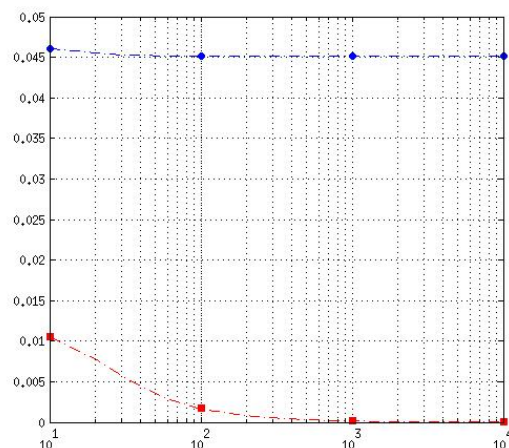
Fig. 2.2. The dependence of the non-oscillatory and oscillatory components of $u(x,t)$ on ω.

2.7. Neumann boundary conditions

If we suppose that the equation (2.1) is given with the Neumann boundary conditions (2.4) in place of the Dirichlet conditions (2.3), the model (2.5) is still correct and equations (2.6) and (2.7) remain valid, except that we need to swap Dirichlet for Neumann conditions in the first. The one difference is that we no longer use Laplace–Dirichlet expansions (2.8) and, fittingly enough, replace them by Laplace–Neumann expansions (also known as *modified Fourier expansions* [21, 25]),

$$f(x) \sim \sum_{m=0}^{\infty} f_m^C \cos \pi m x + \sum_{m=1}^{\infty} f_m^S \sin \pi(m - \tfrac{1}{2})x,$$

where

$$f_m^C = \int_{-1}^{1} f(x) \cos \pi m x \, \mathrm{d}x, \qquad m \in \mathbb{Z}_+,$$

$$f_m^S = \int_{-1}^{1} f(x) \sin \pi(m - \tfrac{1}{2})x \, \mathrm{d}x, \qquad m \in \mathbb{N}.$$

Note that the convergence of Laplace–Neumann expansions is faster than of their Laplace–Dirichlet counterpart: the mth expansion term for $f \in \mathbb{C}^1[-1, 1]$ decays like $\mathcal{O}(m^{-2})$ [25].

The analysis itself is virtually identical to that of the Dirichlet case, hence we will not go into more details here.

2.8. Error Analysis of the Solution

The solution of the univariate diffusion equation (2.1) with a time oscillatory term is given by the infinite expansion (2.14). We need to truncate this expansion in order to obtain the asymptotic-numeric solution of the problem. To perform an error analysis of the solution, we examine the behaviour of the eliminated part of the expansion which we shall henceforth term the remainder. We start with examination of the q_0 term which is given by (2.15). For the sake

of simplicity, we divide the remainder part of this function into six pieces:

$$R_{q_0} = R^{(1)} + R^{(2)} + R^{(3)} + R^{(4)} + R^{(5)} + R^{(6)},$$

where

$$R^{(1)} = \sum_{m=M+1}^{\infty} e^{-\pi^2(m+\frac{1}{2})^2 t} c_m \cos \pi(m+\tfrac{1}{2})x,$$

$$R^{(2)} = \sum_{m=M+1}^{\infty} \left(\int_0^t e^{-\pi^2(m+\frac{1}{2})^2(t-\tau)} \alpha_{0,m}(\tau)\, d\tau \right) \cos \pi(m+\tfrac{1}{2})x,$$

$$R^{(3)} = \sum_{m=0}^{M} \sum_{k=K+1}^{\infty} \frac{(-1)^k \alpha_{n,m}(0)}{[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}} e^{-\pi^2(m+\frac{1}{2})^2 t} \cos \pi(m+\tfrac{1}{2})x$$

$$+ \sum_{m=M+1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{n,m}(0)}{[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}} e^{-\pi^2(m+\frac{1}{2})^2 t} \cos \pi(m+\tfrac{1}{2})x,$$

$$R^{(4)} = \sum_{m=M+1}^{\infty} e^{-\pi^2 m^2 t} d_m \sin \pi m x,$$

$$R^{(5)} = \sum_{m=M+1}^{\infty} \left(\int_0^t e^{-\pi^2 m^2(t-\tau)} \beta_{0,m}(\tau)\, d\tau \right) \sin \pi m x,$$

$$R^{(6)} = \sum_{m=1}^{M} \sum_{k=K+1}^{\infty} \frac{(-1)^k \beta_{n,m}(0)}{[\pi^2 m^2 + in\omega]^{k+1}} e^{-\pi^2 m^2 t} \sin \pi m x$$

$$+ \sum_{m=M+1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \beta_{n,m}(0)}{[\pi^2 m^2 + in\omega]^{k+1}} e^{-\pi^2 m^2 t} \sin \pi m x.$$

We will first take into consideration the term $R^{(1)}$:

$$|R^{(1)}| = \sum_{m=M+1}^{\infty} |e^{-\pi^2(m+\frac{1}{2})^2 t}||c_m|| \cos \pi(m+\tfrac{1}{2})x|.$$

Assume that $|c_m|$ is bounded. In other words, there exists a real number $A \leq \infty$ such that $|c_m| \leq A$. Then, we can write the following inequalities for $R^{(1)}$:

$$\left| R^{(1)} \right| = \sum_{m=M+1}^{\infty} \left| e^{-\pi^2(m+\frac{1}{2})^2 t} \right| |c_m| |\cos \pi(m+\tfrac{1}{2})x|$$

$$\leq \sum_{m=M+1}^{\infty} \left| e^{-\pi^2(m+\frac{1}{2})^2 t} \right| |c_m| \leq A \sum_{m=M+1}^{\infty} \left| e^{-\pi^2(m+\frac{1}{2})^2 t} \right|$$

$$= A \sum_{m=0}^{\infty} \left| e^{-\pi^2(m+M+\frac{3}{2})^2 t} \right|.$$

Secondly, we will find a bound for $R^{(2)}$. We know that

$$|\cos \pi(m+\tfrac{1}{2})x| \leq 1, \quad \text{for} \quad x \in [-1,1].$$

Assume that the *kth* derivative of $\alpha_{n,m}$ is bounded. In other words, there exists a real number $C < \infty$ such that

$$|\alpha_{n,m}^{(k)}(t)| < C, \quad \text{for} \quad t \geq 0.$$

Now, the following inequalities can be written:

$$\left|R^{(2)}\right| \leq \sum_{m=M+1}^{\infty} \left(\int_0^t \left|e^{-\pi^2(m+\frac{1}{2})^2(t-\tau)}\right| |\alpha_{0,m}(\tau)| \, d\tau\right) |\cos \pi(m+\tfrac{1}{2})x|$$

$$\leq C \sum_{m=M+1}^{\infty} \int_0^t \left|e^{-\pi^2(m+\frac{1}{2})^2(t-\tau)}\right| d\tau$$

$$= C \sum_{m=M+1}^{\infty} \left|e^{-\pi^2(m+\frac{1}{2})^2 t} \frac{1}{\pi^2(m+\frac{1}{2})^2} \left(e^{\pi^2(m+\frac{1}{2})t} - 1\right)\right|$$

$$= C \sum_{m=M+1}^{\infty} \left|\frac{1 - e^{-\pi^2(m+\frac{1}{2})^2 t}}{\pi^2(m+\frac{1}{2})^2}\right|$$

$$\leq C \sum_{m=M+1}^{\infty} \left|\frac{1}{\pi^2(m+\frac{1}{2})^2}\right| = C \sum_{m=0}^{\infty} \left|\frac{1}{\pi^2(m+M+\frac{3}{2})^2}\right|.$$

Thirdly, we will find a bound for $R^{(3)}$. For simplicity, we shall analyse the sums separately. Denote the sums as $R_I^{(3)}$ and $R_{II}^{(3)}$ respectively:

$$R_I^{(3)} = \sum_{m=0}^{M} \sum_{k=K+1}^{\infty} \frac{(-1)^k \alpha_{n,m}(0)}{[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}} e^{-\pi^2(m+\frac{1}{2})^2 t} \cos \pi(m+\tfrac{1}{2})x,$$

$$R_{II}^{(3)} = \sum_{m=M+1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_{n,m}(0)}{[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}} e^{-\pi^2(m+\frac{1}{2})^2 t} \cos \pi(m+\tfrac{1}{2})x.$$

Consider the first expansion

$$\left|R_I^{(3)}\right| \leq \sum_{m=0}^{M} \sum_{k=K+1}^{\infty} \frac{|(-1)^k| |\alpha_{n,m}(0)|}{\left|[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}\right|} \left|e^{-\pi^2(m+\frac{1}{2})^2 t}\right| |\cos \pi(m+\tfrac{1}{2})x|$$

$$\leq C \sum_{m=0}^{M} \sum_{k=K+1}^{\infty} \frac{\left|e^{-\pi^2(m+\frac{1}{2})^2 t}\right|}{[\pi^4(m+\frac{1}{2})^4 + n^2\omega^2]^{\frac{k+1}{2}}}$$

$$= C \sum_{m=0}^{M} \sum_{k=0}^{\infty} \frac{\left|e^{-\pi^2(m+\frac{1}{2})^2 t}\right|}{[\pi^4(m+\frac{1}{2})^4 + n^2\omega^2]^{\frac{k+K+2}{2}}}$$

$$= C \sum_{m=0}^{M} \frac{\left|e^{-\pi^2(m+\frac{1}{2})^2 t}\right|}{[\pi^4(m+\frac{1}{2})^4 + n^2\omega^2]^{\frac{K+2}{2}}} \sum_{k=0}^{\infty} \frac{1}{\left(\sqrt{\pi^4(m+\frac{1}{2})^4 + n^2\omega^2}\right)^k}$$

$$= C \sum_{m=0}^{M} \frac{\left|e^{-\pi^2(m+\frac{1}{2})^2 t}\right|}{[\pi^4(m+\frac{1}{2})^4 + n^2\omega^2]^{\frac{K+2}{2}}} \frac{1}{1 - \frac{1}{\sqrt{\pi^4(m+\frac{1}{2})^4 + n^2\omega^2}}}$$

$$\leq \frac{C}{1 - \frac{1}{n\omega}} \sum_{m=0}^{M} \frac{\left|e^{-\pi^2(m+\frac{1}{2})^2 t}\right|}{[\pi^4(m+\frac{1}{2})^4 + n^2\omega^2]^{\frac{K+2}{2}}}.$$

Now, we will find a bound for $R_{II}^{(3)}$. It is possible to write the following inequalities for $R_{II}^{(3)}$:

$$\left|R_{II}^{(3)}\right| \leq \sum_{m=M+1}^{\infty} \sum_{k=0}^{\infty} \frac{|(-1)^k| |\alpha_{n,m}(0)|}{\left|[\pi^2(m+\frac{1}{2})^2 + in\omega]^{k+1}\right|} \left|e^{-\pi^2(m+\frac{1}{2})^2 t}\right| |\cos \pi(m+\tfrac{1}{2})x|$$

$$
\begin{aligned}
&\leq C \sum_{m=M+1}^{\infty} \sum_{k=0}^{\infty} \frac{\left|e^{-\pi^2 (m+\frac{1}{2})^2 t}\right|}{[\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2]^{\frac{k+1}{2}}} \\
&= C \sum_{m=M+1}^{\infty} \sum_{k=0}^{\infty} \frac{\left|e^{-\pi^2 (m+\frac{1}{2})^2 t}\right|}{[\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2]^{\frac{k+1}{2}}} \\
&= C \sum_{m=M+1}^{\infty} \frac{\left|e^{-\pi^2 (m+\frac{1}{2})^2 t}\right|}{[\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2]^{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{1}{\left(\sqrt{\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2}\right)^k} \\
&= C \sum_{m=M+1}^{\infty} \frac{\left|e^{-\pi^2 (m+\frac{1}{2})^2 t}\right|}{[\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2]^{\frac{1}{2}}} \frac{1}{1 - \frac{1}{\sqrt{\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2}}} \\
&\leq \frac{C}{1 - \frac{1}{n\omega}} \sum_{m=M+1}^{\infty} \frac{\left|e^{-\pi^2 (m+\frac{1}{2})^2 t}\right|}{[\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2]^{\frac{1}{2}}} \\
&= \frac{C}{1 - \frac{1}{n\omega}} \sum_{m=0}^{\infty} \frac{\left|e^{-\pi^2 (m+M+\frac{3}{2})^2 t}\right|}{[\pi^4 (m+M+\frac{3}{2})^4 + n^2 \omega^2]^{\frac{1}{2}}}.
\end{aligned}
$$

A similar analysis of the terms $R^{(4)}$, $R^{(5)}$ and $R^{(6)}$ can be done. However, details shall be omitted here. Then, if all of the error terms are assembled, the remainder term for q_0 is obtained as follows:

$$
\begin{aligned}
|R_{q_0}| \leq \sum_{m=0}^{\infty} \Bigg\{ & A\Big[\left|e^{-\pi^2 (m+M+\frac{3}{2})^2 t}\right| + \left|e^{-\pi^2 (m+M+1)^2 t}\right|\Big] \\
& + \frac{C}{\pi^2}\left|\frac{1}{(m+M+\frac{3}{2})^2}\right| + \frac{D}{\pi^2}\left|\frac{1}{(m+M+1)^2}\right| \\
& + \frac{C}{1 - \frac{1}{n\omega}}\left[\frac{\left|e^{-\pi^2 (m+M+\frac{3}{2})^2 t}\right|}{[\pi^4 (m+M+\frac{3}{2})^4 + n^2 \omega^2]^{\frac{1}{2}}}\right] \\
& + \frac{D}{1 - \frac{1}{n\omega}}\left[\frac{\left|e^{-\pi^2 (m+M+1)^2 t}\right|}{[\pi^4 (m+M+1)^4 + n^2 \omega^2]^{\frac{1}{2}}}\right] \Bigg\} \\
& + \frac{C}{1 - \frac{1}{n\omega}} \sum_{m=0}^{M} \frac{\left|e^{-\pi^2 (m+\frac{1}{2})^2 t}\right|}{[\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2]^{\frac{K+2}{2}}} \\
& + \frac{D}{1 - \frac{1}{n\omega}} \sum_{m=0}^{M} \frac{\left|e^{-\pi^2 (m+1)^2 t}\right|}{[\pi^4 (m+1)^4 + n^2 \omega^2]^{\frac{K+2}{2}}},
\end{aligned}
$$

where D is the upper bound of the *kth* derivative of $\beta_{n,m}$. That is, there exists a real number $D < \infty$ such that

$$
|\beta_{n,m}^{(k)}(t)| < D, \quad \text{for} \quad t \geq 0.
$$

If the same procedure is applied to the q_n term, then the remainder term is

$$
\begin{aligned}
|R_{q_n}| \leq \frac{C}{1 - \frac{1}{n\omega}} \sum_{m=0}^{M} \Bigg\{ & \frac{1}{[\pi^4 (m+\frac{1}{2})^4 + n^2 \omega^2]^{\frac{K+2}{2}}} + \frac{1}{[\pi^4 (m+M+\frac{3}{2})^4 + n^2 \omega^2]^{\frac{1}{2}}} \Bigg\} \\
& + \frac{D}{1 - \frac{1}{n\omega}} \sum_{m=0}^{M} \Bigg\{ \frac{1}{[\pi^4 (m+1)^4 + n^2 \omega^2]^{\frac{K+2}{2}}} + \frac{1}{\pi^2 (m+M+1)^2} \Bigg\}.
\end{aligned}
$$

In Fig. 2.3, it is evident that the non-oscillatory component, q_0, does not display any significant dependence on ω. However, in contrast, the oscillatory component, q_n, decays rapidly with ω. The second noticeable feature from Fig. 2.3 is that both components decay rapidly with increasing M.
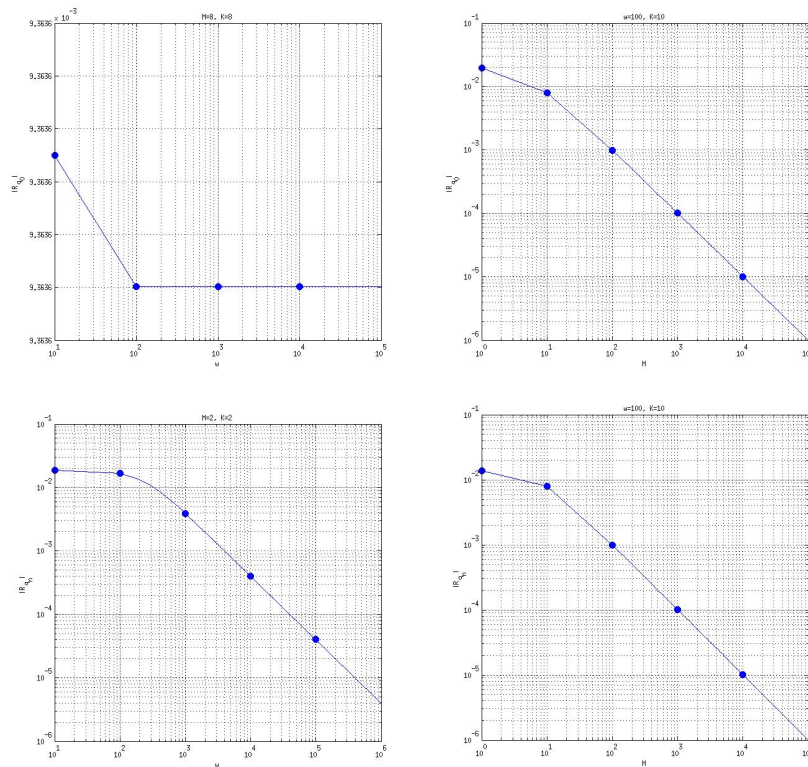

Fig. 2.3. The variation of $\left| R_{q_0} \right|$ and $\left| R_{q_n} \right|$ with ω and M.

3. Space-Like Oscillations

3.1. The general framework

In this section we consider oscillations in space. There are similarities with the narrative of the last section, but also crucial differences. We consider the forced diffusion equation

$$\partial_t u(x,t) = \partial_x^2 u(x,t) + \sum_{n=-\infty}^{\infty} b_n(x,t)\mathrm{e}^{\mathrm{i}n\omega x} \tag{3.1}$$

with the initial condition (2.2) and the Dirichlet boundary conditions (2.3). We present the solution in the form

$$u(x,t) = p_0(x,t) + \sum_{n\neq 0} p_n(x,t,\omega)\mathrm{e}^{\mathrm{i}n\omega x}, \quad t \geq 0, \quad x \in [-1,1], \tag{3.2}$$

analogously to (2.5). Substituting it into (3.1), we have

$$\partial_t p_0(x,t) + \sum_{n \neq 0} \partial_t p_n(x,t,\omega) e^{in\omega x}$$

$$= \partial_x^2 p_0(x,t) + \sum_{n \neq 0} \left[(\partial_x^2 + 2in\omega\partial_x - n^2\omega^2) p_n(x,t,\omega) \right] e^{in\omega x} + \sum_{n=-\infty}^{\infty} b_n(x,t) e^{in\omega x},$$

and separation of frequencies results in equation (2.6) for p_0, while p_n for $n \neq 0$ obeys

$$\partial_t p_n(x,t,\omega) = \left(\partial_x^2 + 2in\omega\partial_x - n^2\omega^2 \right) p_n(x,t,\omega) + b_n(x,t), \quad t \geq 0, \quad x \in [-1,1],$$
$$p_n(x,0) \equiv 0, \quad x \in [-1,1], \tag{3.3}$$
$$p_n(\pm 1, t) \equiv 0, \quad t \geq 0.$$

Thus we can derive p_0 identically to subsection (2.2), using Laplace–Dirichlet expansions.

3.2. Computing p_n

The same methodology is used as in Subsection (2.3) to compute p_n. Substituting

$$b_n(x,t) = \sum_{m=0}^{\infty} \alpha_{n,m}(t) \cos \pi \left(m + \tfrac{1}{2} \right) x + \sum_{m=1}^{\infty} \beta_{n,m}(t) \sin \pi m x, \quad n \neq 0,$$

$$p_n(x,t) = \sum_{m=0}^{\infty} \rho_{n,m}(t) \cos \pi \left(m + \tfrac{1}{2} \right) x + \sum_{m=1}^{\infty} \sigma_{n,m}(t) \sin \pi m x,$$

into (3.3), we obtain the terms

$$-2i\pi n\omega \sum_{m=0}^{\infty} \left(m + \tfrac{1}{2} \right) \rho_{n,m}(t) \sin \pi \left(m + \tfrac{1}{2} \right) x, \quad \text{and} \quad 2\pi i n\omega \sum_{m=1}^{\infty} \sigma_{n,m}(t) \cos \pi m x.$$

Although in principle we can expand $\sin \pi \left(m + \tfrac{1}{2} \right) x$ into Laplace–Dirichlet series in $\sin \pi k x$, $k \in \mathbb{N}$ and $\cos \pi m x$ into a linear combination of $\cos \pi \left(k + \tfrac{1}{2} \right)$, $k \in \mathbb{Z}_+$, this leads to messy expressions and the equations for individual $\rho_{n,m}$s and $\sigma_{n,m}$s are no longer decoupled. Instead, we let

$$\tilde{b}_n(x,t) = e^{in\omega x} b_n(x,t), \quad n \neq 0,$$

and reformulate (3.3) as

$$\partial_t p_n(x,t,\omega) = \left(\partial_x^2 + 2in\omega\partial_x - n^2\omega^2 \right) p_n(x,t,\omega) + e^{-in\omega x} \tilde{b}_n(x,t), \quad t \geq 0, \quad x \in [-1,1],$$
$$p_n(x,0) \equiv 0, \quad x \in [-1,1], \tag{3.4}$$
$$p_n(\pm 1, t) \equiv 0, \quad t \geq 0.$$

Let

$$\tilde{b}_n(x,t) = \sum_{m=0}^{\infty} \tilde{\alpha}_{n,m}(t) \cos \pi \left(m + \tfrac{1}{2} \right) x + \sum_{m=1}^{\infty} \tilde{\beta}_{n,m}(t) \sin \pi m x,$$

where

$$\tilde{\alpha}_{n,m}(t) = \int_{-1}^{1} \tilde{b}_n(x,t) \cos \pi \left(m + \tfrac{1}{2} \right) x \, dx,$$

$$\tilde{\beta}_{n,m}(t) = \int_{-1}^{1} \tilde{b}_n(x,t) \sin \pi m x \, dx.$$

Then direct differentiation confirms that the Laplace–Dirichlet series

$$p_n(x,t) = \mathrm{e}^{-\mathrm{i}n\omega x}\left\{\sum_{m=0}^{\infty}\mathrm{e}^{-\pi^2\left(m+\frac{1}{2}\right)^2 t}\int_0^t \mathrm{e}^{\pi^2\left(m+\frac{1}{2}\right)^2 \tau}\tilde{\alpha}_{n,m}(\tau)\mathrm{d}\tau\cos\pi\left(m+\tfrac{1}{2}\right)x\right.$$
$$\left.+\sum_{m=1}^{\infty}\mathrm{e}^{-\pi^2 m^2 t}\int_0^t \mathrm{e}^{\pi^2 m^2\tau}\tilde{\beta}_{n,m}(\tau)\mathrm{d}\tau\sin\pi m x\right\} \tag{3.5}$$

is the solution of (3.4). (3.5) is in a form amenable to neither analysis nor computation, because of the presence of the highly oscillatory integrals $\tilde{\alpha}_{n,m}$ and $\tilde{\beta}_{n,m}$, and further work is required. The asymptotic expansion can be obtained, but it is encumbered by several disadvantages. In this subsection we propose the alternative of a *Filon-type method* [23, 24]. Within the context of the challenge in hand, we choose $s \in \mathbb{N}$, $\nu \in \mathbb{Z}_+$ and nodes $c_1 < c_2 < \cdots < c_\nu$ in $(-1, 1)$ such that $c_j + c_{\nu+1-j} = 0$. Let $\mathcal{P} : C^s[-1, 1] \to \mathbb{P}_{\nu+2s-1}$ take each f into its Hermite interpolation polynomial,

$$\frac{\mathrm{d}\mathcal{P}[f](-1)}{\mathrm{d}x^k} = f^{(k)}(-1), \quad \frac{\mathrm{d}\mathcal{P}[f](1)}{\mathrm{d}x^k} = f^{(k)}(1), \quad k = 0, 1, \cdots, s-1,$$
$$\mathcal{P}[f](c_j) = f(c_j), \quad j = 1, 2, \cdots, \nu.$$

Then

$$\int_{-1}^{1} f(x)\mathrm{e}^{\eta x}\mathrm{d}x \approx \mathcal{F}_\eta[f] = \int_{-1}^{1}\mathcal{P}[f](x)\mathrm{e}^{\eta x}\mathrm{d}x. \tag{3.6}$$

It follows from (2.11) that

$$\mathcal{F}_\eta[f] = \mathrm{e}^\eta \sum_{k=0}^{s}\frac{(-1)^k}{\eta^{k+1}}\frac{\mathrm{d}^k\mathcal{P}[f](1)}{\mathrm{d}x^k} - \mathrm{e}^{-\eta}\sum_{k=0}^{s}\frac{(-1)^k}{\eta^{k+1}}\frac{\mathrm{d}^k\mathcal{P}[f](-1)}{\mathrm{d}x^k}. \tag{3.7}$$

Using the same method of proof as in [23] , we can show that for any $\eta \in \mathbb{C}$, Re $\eta \le 0$, it is true that

$$\int_{-1}^{1} f(x)\mathrm{e}^{\eta x}\mathrm{d}x \sim \mathrm{F}_\eta[f] + \mathcal{O}(\eta^{-s-2}), \quad |\eta| \gg 1. \tag{3.8}$$

However, unlike the asymptotic expansion, the Filon-type method produces a quality numerical solution also for small η : as $\eta \to 0$, it reduces to classical interpolatory quadrature of Birkhoff–Hermite type at the requisite nodes (both the c_js and ± 1, the latter of multiplicity s). In other words, (3.6) is uniformly good for all Re $\eta \ge 0$.

The choice of the internal nodes $c_1, \cdots, c_\nu$ is crucial to ensure that the method performs well for small η. Clearly, in that case we should maximise classical quadrature order, and this occurs once the c_is are the zeros of the Jacobi polynomial $P_\nu^{(s,s)}$ [23]. Note that such zeros are indeed symmetric with respect to the origin.

We approximate

$$\tilde{\alpha}_{n,m} \approx \tfrac{1}{2}\left\{\mathcal{F}_{n\omega+\pi\left(m+\frac{1}{2}\right)}[b_n] + \mathcal{F}_{n\omega-\pi\left(m+\frac{1}{2}\right)}[b_n]\right\}, \quad n \ne 0, \quad m \in \mathbb{Z}_+, \tag{3.9a}$$
$$\tilde{\beta}_{n,m} \approx \tfrac{1}{2\mathrm{i}}\left\{\mathcal{F}_{n\omega+\pi m}[b_n] - \mathcal{F}_{n\omega-\pi m}[b_n]\right\}, \quad n \ne 0, \quad m \in \mathbb{N}. \tag{3.9b}$$

It is helpful to represent the Filon-type method (3.6) explicitly as a linear combination of function values and derivatives, rather than in the form (3.7). Thus, let $l_1, \cdots, l_\nu, p_0, \cdots, p_{s-1} \in \mathbb{P}_{\mu+2s+1}$ be such that

$$l_k(c_i) = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases}, \qquad i = 1, \cdots, \nu,$$

$$l_k^{(j)}(\pm 1) = 0, \quad j = 0, \cdots, s-1,$$

$$p_j(c_i) = 0, \qquad i = 1, \cdots, \nu,$$

$$p_j^{(k)}(-1) = 0, \quad p_j^{(k)}(1) = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad k = 0, \cdots, s-1.$$

(Note that $l_{\nu+1-j}(x) = l_j(-x)$.) Then

$$\mathcal{P}[f](x) = \sum_{k=1}^{\nu} l_k(x) f(c_k) + \sum_{j=0}^{s-1} \left[p_j(x) f^{(j)}(1) + (-1)^j p_j(-x) f^{(j)}(-1) \right]. \tag{3.10}$$

Therefore,

$$\mathcal{F}_\eta[f] = \sum_{k=1}^{\nu} d_k(\eta) f(c_k) + \sum_{j=0}^{s-1} \left[e_j(\eta) f^{(j)}(1) + (-1)^j e_j(-\eta) f^{(j)}(-1) \right], \tag{3.11}$$

where

$$d_k(\eta) = \int_{-1}^{1} l_k(x) e^{\eta x} dx, \quad k = 1, \cdots, \nu, \tag{3.12a}$$

$$e_j(\eta) = \int_{-1}^{1} p_j(x) e^{\eta x} dx, \quad j = 0, \cdots, s-1. \tag{3.12b}$$

We compute the variable weights d_k and e_j explicitly and use (3.12) to compute the $\tilde{\alpha}_{n,m}$s and the $\tilde{\beta}_{n,m}$s. Note that

$$d_k(\eta) = e^\eta d_k^+(\eta) - e^{-\eta} d_k^-(\eta), \tag{3.13a}$$

$$e_j(\eta) = e^\eta e_j^+(\eta) - e^{-\eta} e_j^-(\eta), \tag{3.13b}$$

where $d_k^\pm$ and $e_j^\pm$ are polynomials in η^{-1},

$$d_k^\pm(\eta) = \sum_{i=0}^{2s+\nu-1} \frac{(-1)^i}{\eta^{i+1}} l_k^{(i)}(\pm 1), \quad e_j^\pm(\eta) = \sum_{i=0}^{2s+\nu-1} \frac{(-1)^i}{\eta^{i+1}} p_k^{(i)}(\pm 1). \tag{3.14}$$

If we suppose that equation (3.1) is given with Neumann boundary conditions, we use Laplace–Neumann expansions in order to solve these equations. We make an identical analysis to that of the Dirichlet case and obtain the solution of (3.1).

3.3. A worked-out example for space-like oscillations

In this section, we present a straightforward example that illustrates the merits of the method introduced above. It considers a forcing term that is oscillatory in space.

$$\partial_t u = \partial_x^2 u + b \sin \omega x, \tag{3.15}$$

with the initial and boundary conditions

$$u(x, 0) = \phi(x) = \cos \omega x,$$

$$u(-1, t) = \nu_-(t) = e^{-\omega^2 t} \cos \omega - \frac{b}{\omega^2} \left(1 - e^{-\omega^2 t}\right) \sin \omega,$$

$$u(1, t) = \nu_+(t) = e^{-\omega^2 t} \cos \omega + \frac{b}{\omega^2} \left(1 - e^{-\omega^2 t}\right) \sin \omega.$$

Its solution takes the form

$$u(x, t) = p_0(x, t) + \left[e^{-i\omega x} p_{-1}(x, t, \omega) + e^{i\omega x} p_1(x, t, \omega)\right].$$

To determine $p_0(x, t)$, the non-oscillatory differential equation (2.6) is considered with the initial condition

$$p_0(x, 0) = \phi(x) = \cos \omega x, \qquad x \in [-1, 1],$$

and the Dirichlet boundary conditions

$$p_0(-1, t) = \nu_-(t) = e^{-\omega^2 t} \cos \omega - \frac{b}{\omega^2} \left(1 - e^{-\omega^2 t}\right) \sin \omega,$$

$$p_0(1, t) = \nu_+(t) = e^{-\omega^2 t} \cos \omega + \frac{b}{\omega^2} \left(1 - e^{-\omega^2 t}\right) \sin \omega, \qquad t \geq 0.$$

The function

$$\tilde{p}_0(x, t) = p_0(x, t) - e^{-\omega^2 t} \cos \omega - x \frac{b}{\omega^2} \left(1 - e^{-\omega^2 t}\right) \sin \omega,$$

obeys the partial difference equation.

$$\partial_t \tilde{p}_0(x, t) = \partial_x^2 \tilde{p}_0(x, t) + \omega^2 e^{-\omega^2 t} \cos \omega - xb e^{-\omega^2 t} \sin \omega,$$

while the initial condition and the boundary conditions take the form

$$\tilde{p}_0(x, 0) = \phi(x) - \cos \omega = \cos \omega x - \cos \omega, \qquad x \in [-1, 1],$$
$$\tilde{p}_0(\pm 1, t) = 0, \qquad t \geq 0.$$

This yields $\tilde{\phi}$ and $\tilde{b}_0$, namely

$$\tilde{\phi}(x) = \cos \omega x - \cos \omega,$$
$$\tilde{b}_0(x, t) = \omega^2 e^{-\omega^2 t} \cos \omega - xb e^{-\omega^2 t} \sin \omega.$$

These functions are used to compute $\alpha_{0,m}$, $\beta_{0,m}$ and c_m, d_m, and this results in $\rho_{0,m}$ and $\sigma_{0,m}$,

$$\rho_{0,m}(t) = e^{-\pi^2 \left(m + \frac{1}{2}\right)^2 t} c_m + \frac{4(-1)^m \omega^2 \cos \omega}{\pi(2m+1)\left(\pi^2 \left(m + \frac{1}{2}\right)^2 - \omega^2\right)} \left[e^{-\omega^2 t} - e^{-\pi^2 \left(m + \frac{1}{2}\right)^2 t}\right],$$

$$\sigma_{0,m}(t) = e^{-\pi^2 m^2 t} d_m + \frac{2b(-1)^m \pi m \sin \omega}{\pi^2 m \left(\pi^2 m^2 - \omega^2\right)} \left(e^{-\omega^2 t} - e^{-\pi^2 m^2 t}\right).$$

In the next step we compute p_{-1} and p_1 by solving the differential equations,

$$\partial_t p_{-1} = \left(\partial_x^2 - 2i\omega \partial_x - \omega^2\right) p_{-1} + e^{i\omega x} \tilde{b}_{-1},$$
$$\partial_t p_1 = \left(\partial_x^2 + 2i\omega \partial_x - \omega^2\right) p_1 + e^{-i\omega x} \tilde{b}_1,$$
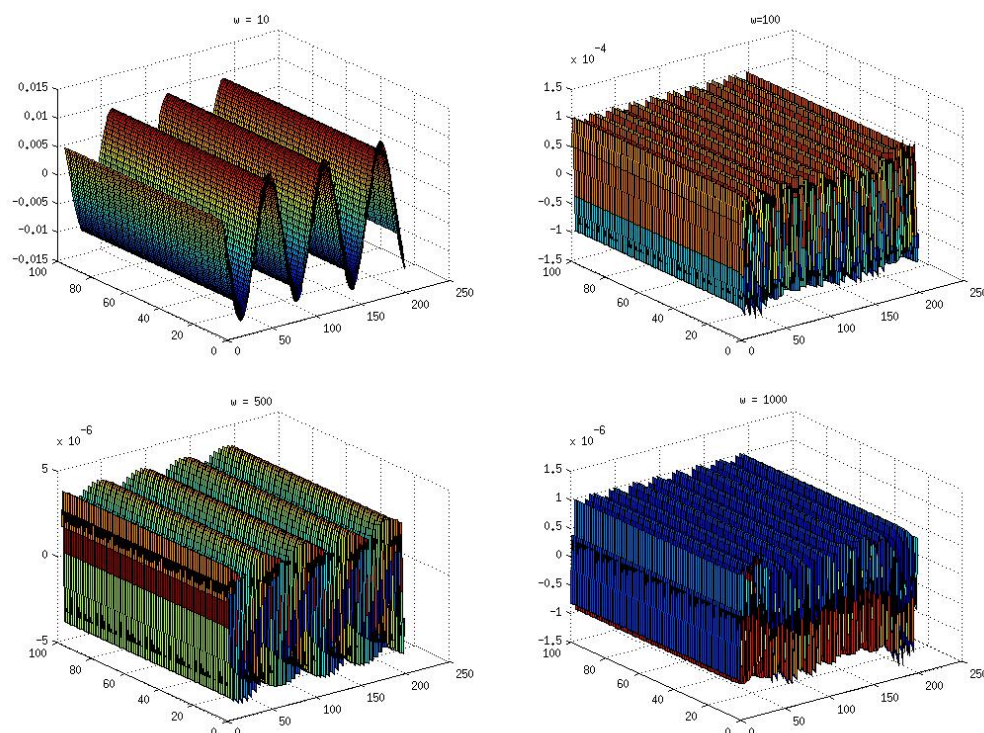
Fig. 3.1. *The non-oscillatory (on the left) and oscillatory components of $u(x,t)$, the solution of (3.15), for different values of ω.*

where $\tilde{b}_{-1}$ and $\tilde{b}_1$ are given by

$$\tilde{b}_{-1}(x,t) = \frac{b}{2}\mathrm{i}, \qquad \tilde{b}_1(x,t) = -\frac{b}{2}\mathrm{i}.$$

The initial and boundary conditions are

$$p_{\pm 1}(x,0) \equiv 0, \quad x \in [-1,1], \qquad p_{\pm 1}(\pm 1, t) \equiv 0, \quad t \geq 0.$$

Using (3.5), we thus obtain

$$
p_{-1}(x,t) = \mathrm{e}^{\mathrm{i}\omega x}\left[\sum_{m=0}^{\infty} \mathrm{e}^{-\pi^2(m+\frac{1}{2})^2 t}\int_0^t \mathrm{e}^{\pi^2(m+\frac{1}{2})^2\tau}\tilde{\alpha}_{-1,m}(\tau)\mathrm{d}\tau\cos\pi(m+\tfrac{1}{2})x\right.
$$
$$
\left. + \sum_{m=1}^{\infty}\mathrm{e}^{-\pi^2 m^2 t}\int_0^t e^{\pi^2 m^2\tau}\tilde{\beta}_{-1,m}(\tau)\mathrm{d}\tau\sin\pi m x\right],
$$
$$
p_1(x,t) = \mathrm{e}^{-\mathrm{i}\omega x}\left[\sum_{m=0}^{\infty}\mathrm{e}^{-\pi^2(m+\frac{1}{2})^2 t}\int_0^t \mathrm{e}^{\pi^2(m+\frac{1}{2})^2\tau}\tilde{\alpha}_{1,m}(\tau)\mathrm{d}\tau\cos\pi(m+\tfrac{1}{2})x\right.
$$
$$
\left. + \sum_{m=0}^{\infty}\mathrm{e}^{-\pi^2 m^2 t}\int_0^t e^{\pi^2 m^2\tau}\tilde{\beta}_{1,m}(\tau)\mathrm{d}\tau\sin\pi m x\right],
$$

where $\tilde{\alpha}_{-1,m}$, $\tilde{\alpha}_{1,m}$, $\tilde{\beta}_{-1,m}$ and $\tilde{\beta}_{1,m}$ can be evaluated either analytically or numerically, using (3.9).
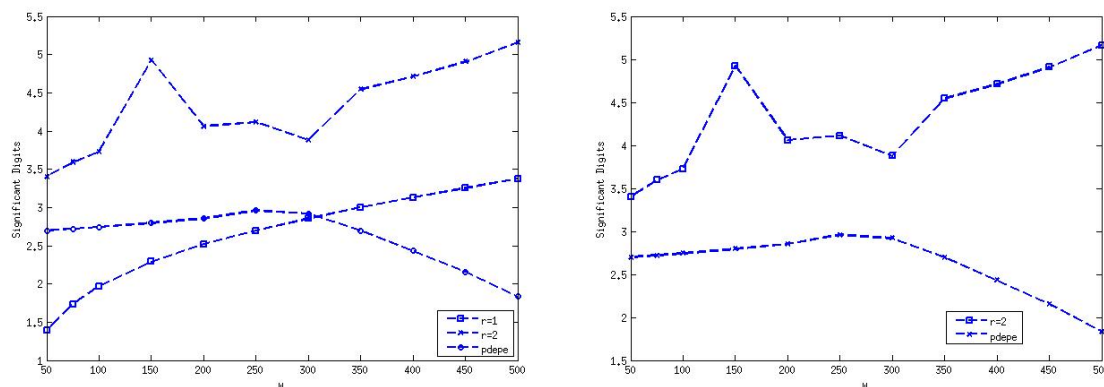
Fig. 3.2. The number of significant digits of the L_∞ error (for $x \in [-1, 1]$ and $t \in [0, 1]$) for different values of ω.

In Fig. 3.2, we display the number of significant digits attained by our method, implemented with $b = 1$, comparing it with the `pdepe` function of MATLAB (which uses a second-order finite-difference spatial discretization, followed by an ODE solver). The latter is executed with the tolerance set at 10^{-10}. Note that already for $r = 2$ the error committed with the new method is significantly less than that of the `pdepe` subroutine as frequency increases. Needless to say, the execution time with our method is significantly smaller than with `pdepe`.

4. Conclusion

This is an initial investigation into a new subject matter, in a nature of a feasibility study. Practical applications to specific problems would require much further work, this is typical to most applications of numerical PDEs. For example, the application which motivated us is an excitation of an antenna by highly oscillatory current. However, realistic modelling of this phenomenon would require the presence of ODE, PDE and indeed DAE components. We believe that at the first instance each of these individual problems must be addressed separately and, as evident in the present paper, this is neither trivial nor brief.

Appendix A

We present the calculations of c_m, d_m, $\alpha_{n,m}^{(k)}$ and $\beta_{n,m}^{(k)}$ as referred to in Section 2.6.

$$c_m = \int_{-1}^{1} (\cos 2\pi x - 1) \cos \pi (m + \tfrac{1}{2}) x \, \mathrm{d}x = \frac{8(-1)^m}{\pi (m - \tfrac{3}{2})(m + \tfrac{1}{2})(m + \tfrac{5}{2})},$$

$$d_m = \int_{-1}^{1} (\cos 2\pi x - 1) \sin \pi m x \, \mathrm{d}x = 0,$$

$$\alpha_{0,m} = \pi^2 e^{-\pi^2 t} \int_{-1}^1 \cos \pi (m + \tfrac{1}{2}) x \, dx = \frac{2(-1)^m \pi e^{-\pi^2 t}}{(m + \tfrac{1}{2})},$$

$$\beta_{0,m} = \pi^2 e^{-\pi^2 t} \int_{-1}^1 \sin m\pi x \, dx \equiv 0,$$

$$\alpha^{(k)}_{\pm 1,m}(t) = \frac{(-\pi^2)^k}{2} e^{-\pi^2 t} \int_{-1}^1 \cos \pi x \cos \pi (m + \tfrac{1}{2}) x \, dx = \frac{(-1)^{k+m+1} \pi^{2k} e^{-\pi^2 t}(m + \tfrac{1}{2})}{\pi(m - \tfrac{1}{2})(m + \tfrac{3}{2})},$$

$$\beta^{(k)}_{\pm 1,m}(t) = \frac{(-\pi^2)^k}{2} e^{-\pi^2 t} \int_{-1}^1 \cos \pi x \sin \pi m x \, dx \equiv 0.$$

Trivially, $\alpha_{n,m}, \beta_{n,m} \equiv 0$ for $|n| \geq 2$. Finally,

$$\int_0^t e^{\pi^2 (m + \frac{1}{2})^2 \tau} \alpha_{0,m}(\tau) \, d\tau = \frac{2(-1)^m}{\pi(m - \tfrac{1}{2})(m + \tfrac{1}{2})(m + \tfrac{3}{2})} \left[e^{\pi^2 (m - \frac{1}{2})(m + \frac{3}{2})t} - 1 \right],$$

$$\int_0^t e^{\pi^2 m^2 \tau} \beta_{0,m}(\tau) \, d\tau \equiv 0.$$

Substituting into (2.15) and summing geometric series, we thus have

$$
\begin{aligned}
q_0(x,t) &\sim e^{-\pi^2 t} + \sum_{m=0}^\infty e^{-\pi^2 (m + \frac{1}{2})^2 t} \Bigg\{ \frac{8(-1)^m}{\pi(m - \tfrac{3}{2})(m + \tfrac{1}{2})(m + \tfrac{5}{2})} \\
&\quad + \frac{2(-1)^m}{\pi^3 (m - \tfrac{1}{2})(m + \tfrac{1}{2})(m + \tfrac{3}{2})} \left[e^{\pi^2 (m - \frac{1}{2})(m + \frac{3}{2})t} - 1 \right] \\
&\quad - \frac{(-1)^{m+1}(m + \tfrac{1}{2}) e^{-\pi^2 t}}{\pi(m - \tfrac{1}{2})(m + \tfrac{3}{2})} \sum_{k=0}^\infty \frac{\pi^{2k}}{[\pi^2 (m + \tfrac{1}{2})^2 + i\omega]^{k+1}} \\
&\quad - \frac{(-1)^{m+1}(m + \tfrac{1}{2}) e^{-\pi^2 t}}{\pi(m - \tfrac{1}{2})(m + \tfrac{3}{2})} \sum_{k=0}^\infty \frac{\pi^{2k}}{[\pi^2 (m + \tfrac{1}{2})^2 - i\omega]^{k+1}} \Bigg\} \cos \pi (m + \tfrac{1}{2}) x \\
&= e^{-\pi^2 t} \left[1 + \frac{2}{\pi} \sum_{m=0}^\infty \frac{(-1)^m}{(m - \tfrac{1}{2})(m + \tfrac{1}{2})(m + \tfrac{3}{2})} \cos \pi (m + \tfrac{1}{2}) x \right] \\
&\quad + \sum_{m=0}^\infty (-1)^m e^{-\pi^2 (m + \frac{1}{2})^2 t} \left[\frac{8}{\pi(m - \tfrac{3}{2})(m + \tfrac{1}{2})(m + \tfrac{5}{2})} \right. \\
&\quad - \frac{2}{\pi(m - \tfrac{1}{2})(m + \tfrac{1}{2})(m + \tfrac{3}{2})} + \frac{(m + \tfrac{1}{2}) e^{-\pi^2 t}}{\pi(m - \tfrac{1}{2})(m + \tfrac{3}{2})} \cdot \frac{1}{\pi^2 (m - \tfrac{1}{2})(m + \tfrac{3}{2}) + i\omega} \\
&\quad \left. + \frac{(m + \tfrac{1}{2}) e^{-\pi^2 t}}{\pi(m - \tfrac{1}{2})(m + \tfrac{3}{2})} \cdot \frac{1}{\pi^2 (m - \tfrac{1}{2})(m + \tfrac{3}{2}) - i\omega} \right] \cos \pi (m + \tfrac{1}{2}) x \\
&= e^{-\pi^2 t} \left[1 + \frac{2}{\pi} \sum_{m=0}^\infty \frac{(-1)^m}{(m - \tfrac{1}{2})(m + \tfrac{1}{2})(m + \tfrac{3}{2})} \cos \pi (m + \tfrac{1}{2}) x \right] \\
&\quad + 2 \sum_{m=0}^\infty (-1)^m e^{-\pi^2 (m + \frac{1}{2})^2 t} \left[\frac{4}{\pi(m - \tfrac{3}{2})(m + \tfrac{1}{2})(m + \tfrac{5}{2})} \right. \\
&\quad \left. - \frac{1}{\pi(m - \tfrac{1}{2})(m + \tfrac{1}{2})(m + \tfrac{3}{2})} - \frac{2\pi(m + \tfrac{1}{2}) e^{-\pi^2 t}}{\pi^4 (m - \tfrac{1}{2})^2 (m + \tfrac{3}{2})^2 + \omega^2} \right] \cos \pi (m + \tfrac{1}{2}) x.
\end{aligned}
$$

This can be further simplified, because

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{(m - \frac{1}{2})(m + \frac{1}{2})(m + \frac{3}{2})} \cos \pi(m + \tfrac{1}{2})x = -\pi \cos^2 \frac{\pi x}{2},$$

therefore the first set of square brackets can be replaced by $\mathrm{e}^{-\pi^2 t} \cos \pi x$. Likewise,

$$\begin{aligned}
q_{\pm 1}(x, t) &\sim \mathrm{e}^{-\pi^2 t} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(m + \frac{1}{2})}{\pi(m - \frac{1}{2})(m + \frac{3}{2})} \sum_{k=0}^{\infty} \frac{1}{[\pi^2(m + \frac{1}{2})^2 \pm \mathrm{i}\omega]^{k+1}} \\
&= -\mathrm{e}^{-\pi^2 t} \sum_{m=0}^{\infty} \frac{(-1)^m(m + \frac{1}{2})}{\pi(m - \frac{1}{2})(m + \frac{3}{2})} \cdot \frac{\cos \pi(m + \frac{1}{2})x}{\pi^2(m - \frac{1}{2})(m + \frac{3}{2}) \pm \mathrm{i}\omega},
\end{aligned}$$

therefore

$$\begin{aligned}
\sum_{n \neq 0} q_n(x, t)\mathrm{e}^{\mathrm{i}n\omega t} &\sim -\mathrm{e}^{-\pi^2 t} \sum_{m=0}^{\infty} \frac{(-1)^m(m + \frac{1}{2})}{\pi(m - \frac{1}{2})(m + \frac{3}{2})} \left[\frac{\mathrm{e}^{\mathrm{i}\omega t}}{\pi^2(m - \frac{1}{2})(m + \frac{3}{2}) + \mathrm{i}\omega} \right. \\
&\quad \left. + \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\pi^2(m - \frac{1}{2})(m + \frac{3}{2}) - \mathrm{i}\omega} \right] \cos \pi(m + \tfrac{1}{2})x \\
&= -2\pi \mathrm{e}^{-\pi^2 t} \cos \omega t \sum_{m=0}^{\infty} \frac{(-1)^m(m + \frac{1}{2})}{\pi^4(m - \frac{1}{2})^2(m + \frac{3}{2})^2 + \omega^2} \cos \pi(m + \tfrac{1}{2})x \\
&\quad - 2\omega \mathrm{e}^{-\pi^2 t} \sin \omega t \sum_{m=0}^{\infty} \frac{(-1)^m(m + \frac{1}{2})}{\pi(m - \frac{1}{2})(m + \frac{3}{2})} \cdot \frac{\cos \pi(m + \frac{1}{2})x}{\pi^4(m - \frac{1}{2})^2(m + \frac{3}{2})^2 + \omega^2}.
\end{aligned}$$

References

[1] B.J. Adcock, Modified Fourier Expansions: Theory, Construction and Applications, PhD Thesis, University of Cambridge, 2010.

[2] B.J. Adcock and A.Iserles and S.P.Nørsett, From high oscillation to rapid approximation II: Expansions in Birkhoff Series, *IMA J. Num. Anal.*, (2011).

[3] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications Inc., 1964.

[4] J.P. Boyd, Chebyshev and Fourier Spectral Methods, Springer Verlag, 1989.

[5] E.O. Brigham, The Fast Fourier Transform Prentice-Hall, Englewood Cliffs, NJ, 1974.

[6] D. Cohen and E. Hairer and C. Lubich, Modulated Fourier expansions of highly oscillatory differential equations, *Found. Comp. Maths.*, **3** (2005), 327-450.

[7] D. Cohen and E. Hairer and C. Lubich, Conservation of Energy, momentum and actions in numerical discretizations of nonlinear wave equations, *Numer. Math.*, **110** (2008), 113-143.

[8] D. Cohen and E. Hairer and C. Lubich, Long-time analysis of nonlinearly perturbed wave equations via modulated Fourier expansions, *Arch. Ration. Mech. Anal.*, **187** (2008), 341-368.

[9] E. Faou and L. Gauckler and C. Lubich, Sobolev stability of plane wave solutions to the cubic nonlinear Schrödinger equation on a torus, in press.

[10] L. Gauckler and C. Lubich, Nonlinear Schrödinger equations and their spectral semi-discretizations over long times, *Found. Comput. Math.*, **10** (2010), 275-302.

[11] E. Hairer and C. Lubich, Spectral semi-discretizations of weakly nonlinear wave equations over long times, *Found. Comput. Math.*, **8** (2008), 319-334.

[12] M. Condon and A. Deaño and A. Iserles, On highly oscillatory problems arising in electronic engineering, *M2AN*, **43** (2009), 785-804.

[13] M. Condon and A. Deaño and A. Iserles and K. Maczynski and T. Xu, On Numerical Methods for Highly Oscillatory Problems in Circuit Simulation, *COMPEL*, **28** (2009), 1607-1618.

[14] M. Condon and A. Deaño and A. Iserles, On Systems of Differential Equations with Extrinsic Oscillation, *Discr. and Cont. Dynamical Sys.*, **28** (2010), 1345-1367.

[15] M. Condon and A. Deaño and A. Iserles, On second order differential equations with highly oscillatory forcing terms, *Proc. Royal Soc. A.*, **466** (2010), 1809-1828.

[16] D. Huybrechs and A. Iserles and S.P. Nørsett, From high oscillation to rapid approximation V: the equilateral triangle, *IMA J. Numer. Anal.* , **31**:3 (2011), 755-785.

[17] B. Fornberg, A Practical Guide to Pseudospectral Methods, Cambridge University Press, Cambridge, UK, 1998.

[18] E. Hairer and C. Lubich and G. Wanner, Geometric Numeric Integration, Springer Verlag, 2006.

[19] J.S. Hesthaven and S. Gottlieb and D. Gottlieb, Spectral Methods for Time-Dependent Problems, Cambridge University Press, Cambridge, UK, 2007.

[20] D. Huybrechs and S. Olver, Highly oscillatory quadrature, in *London Mathematical Society Lecture Note Series: Highly Oscillatory Problems*, B. Engquist and A. Fokas and E. Hairer and A. Iserles (ed.), Cambridge University Press, Cambridge, UK, **366** (2009), 25-50.

[21] D. Huybrechs and S. Olver, Rapid function approximation by modified Fourier series, in *London Mathematical Society Lecture Note Series: Highly Oscillatory Problems*, B. Engquist and A. Fokas and E. Hairer and A. Iserles (ed.), Cambridge University Press, Cambridge, UK, **366** (2009), 51-71.

[22] A. Iserles, A First Course in the Numerical Analysis of Differential Equations, Cambridge University Press, Cambridge, UK, 2009.

[23] A. Iserles and S.P. Nørsett, Efficient quadrature of highly oscillatory integrals using derivatives, *Proc. Royal Soc. A*, **461** (2005), 1383-1399.

[24] A. Iserles and S.P. Nørsett, On quadrature methods for highly oscillatory integrals and their implementation, *BIT*, **44** (2004), 755-772.

[25] A. Iserles and S.P. Nørsett, From high oscillation to rapid approximation I: Modified Fourier expansions, *IMA J. Num. Anal.*, **28** (2008), 862-887.

[26] F. John, Partial Differential Equations, Springer-Verlag, New York, USA, 1982.

[27] M.A. Naimark, Linear Differential Operators, Harrap, London, 1968.

[28] J.F. Oliveira and J.C. Pedro, An efficient time-domain simulation method for multirate RF nonlinear circuits, *IEEE Trans. on Microwave Theory and Tech.*, **55** (2007), 2384-2392.

[29] R. Pulch and M. Günther, A method of characteristics for solving multirate partial differential equations in radio frequency application, *Appld Num. Maths*, **42** (2002), 397-409.