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ON BLOCK PRECONDITIONERS FOR PDE-CONSTRAINED OPTIMIZATION PROBLEMS*

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Abstract

Recently, Bai proposed a block-counter-diagonal and a block-counter-triangular preconditioning matrices to precondition the GMRES method for solving the structured system of linear equations arising from the Galerkin finite-element discretizations of the distributed control problems in (Computing 91 (2011) 379-395). He analyzed the spectral properties and derived explicit expressions of the eigenvalues and eigenvectors of the preconditioned matrices. By applying the special structures and properties of the eigenvector matrices of the preconditioned matrices, we derive upper bounds for the 2-norm condition numbers of the eigenvector matrices and give asymptotic convergence factors of the preconditioned GMRES methods with the block-counter-diagonal and the block-counter-triangular preconditioners. Experimental results show that the convergence analyses match well with the numerical results.

Mathematics subject classification: 65F08, 65F10.
Key words: PDE-constrained optimization, GMRES method, Preconditioner, Condition number, Asymptotic convergence factor.

1. Introduction

Preconditioning technique as an efficient tool has been widely applied in Krylov subspace methods for solving linear systems arising from discretizations of partial differential equations. In [3], Bai considered using the preconditioned Krylov subspace methods to solve the linear system emerging from the following distributed control problem

$$\min_{u,f} \frac{1}{2}\|u-u_*\|_2^2 + \beta\|f\|_2^2, \tag{1.1}$$

$$\text{subject to} \quad -\nabla^2 u = f \quad \text{in } \Omega, \tag{1.2}$$

$$\text{with} \quad u = g \quad \text{on} \quad \partial\Omega_1 \quad \text{and} \quad \frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega_2, \tag{1.3}$$

where the domain $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$, $\partial\Omega_1$ and $\partial\Omega_2$ are distinct, $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$ and $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, u_* is the known desired state. This problem was first introduced by Lions in [10]. We need to find u which satisfies the PDE problem (1.1)-(1.3) and is as close to u_* as possible in L_2-norm sense. A recent reference on this topic can be found in [9].

By adopting the discretize-then-optimize approach and employing the Galerkin finite element method in the discretization, the PDE-constrained optimization problem (1.1)-(1.3) can be transformed into a discrete analogue of the minimization problem. By applying the Lagrange

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multiplier technique to the minimization problem, we find that $\mathbf{f}$ and $\mathbf{u}$ can be defined by the following linear system

$$\boldsymbol{A}\mathbf{x} \equiv \begin{pmatrix} 2\beta \boldsymbol{M} & \mathbf{0} & -\boldsymbol{M} \\ \mathbf{0} & \boldsymbol{M} & \boldsymbol{K}^T \\ -\boldsymbol{M} & \boldsymbol{K} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{f} \\ \mathbf{u} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{b} \\ \mathbf{d} \end{pmatrix} \equiv \mathbf{g}, \tag{1.4}$$

where $\boldsymbol{M} \in \mathbb{R}^{m\times m}$ is the symmetric positive definite mass matrix, $\boldsymbol{K} \in \mathbb{R}^{m\times m}$ is the symmetric stiffness matrix (the discrete Laplacian), $\mathbf{d} \in \mathbb{R}^m$ contains the terms coming from the boundary values of the discrete solution, $\mathbf{b} \in \mathbb{R}^m$ is the Galerkin projection of the desired state u_* and λ is a vector of Lagrange multipliers, see also [7]. (1.4) is a saddle point problem if we write it in a 2-by-2 block form, see, for instance, [1, 2, 6]. Due to the finite element discretization, $\boldsymbol{M}$ and $\boldsymbol{K}$ are very large and sparse, the matrix $\boldsymbol{A}$ is large and sparse, too. By making use of the easiness of matrix-vector multiplications and linear computation in Krylov subspace methods, many preconditioned Krylov subspace methods have been proposed for solving (1.4), see, for instance, [3, 5, 8, 11–13]. Specifically, Bai applied the preconditioned GMRES method to solve the system (1.4) in [3]. He introduced two efficient preconditioners $\boldsymbol{P_{BCD}}$ and $\boldsymbol{P_{BCT}}$ to accelerate convergence rates of the GMRES method. $\boldsymbol{P_{BCD}}$ is a block-counter-diagonal preconditioner of form

$$\boldsymbol{P_{BCD}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & -\boldsymbol{M} \\ \mathbf{0} & \boldsymbol{M} & \mathbf{0} \\ -\boldsymbol{M} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{1.5}$$

and $\boldsymbol{P_{BCT}}$ is a block-counter-triangular preconditioner of form

$$\boldsymbol{P_{BCT}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & -\boldsymbol{M} \\ \mathbf{0} & \boldsymbol{M} & \boldsymbol{K}^T \\ -\boldsymbol{M} & \boldsymbol{K} & \mathbf{0} \end{pmatrix}. \tag{1.6}$$

It is clearly to see that the computation of $\boldsymbol{P_{BCD}}$ or $\boldsymbol{P_{BCT}}$ only requires to solve three linear sub-systems with the same coefficient matrix $\boldsymbol{M}$, and does not need to solve any linear sub-system with coefficient matrix $\boldsymbol{K}$. Therefore, the implementations of the preconditioned GMRES methods with these preconditioners for (1.4) are easy and effective.

In [3], the author also gave the spectral properties of the preconditioned matrices $\boldsymbol{P_{BCD}^{-1}A}$ and $\boldsymbol{P_{BCT}^{-1}A}$.

Theorem 1.1. (Theorem 2.1 in [3]) *Let $\boldsymbol{A} \in \mathbb{R}^{3m\times 3m}$ be the coefficient matrix of the saddle-point problem* (1.4) *and $\boldsymbol{P_{BCD}} \in \mathbb{R}^{3m\times 3m}$ be the block-counter-diagonal preconditioner of $\boldsymbol{A}$ defined in* (1.5)*. Assume that v_l is an eigenvalue and $\mathbf{x}^{(l)} \in \mathbb{C}^m$ is the corresponding eigenvector of the matrix $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T \in \mathbb{R}^{m\times m}$, $l = 1, \ldots, m$, where $v_l > 0$ $(l = 1, \ldots, m)$. Then*

1. *the eigenvalues of the preconditioned matrix $\boldsymbol{P_{BCD}^{-1}A}$ are*

$$\lambda_k^{(l)} := 1 - \sqrt[3]{2\beta v_l}\, e^{\frac{(2k+1)\pi\imath}{3}}, \quad k = 0, 1, 2,\ l = 1, \ldots, m,$$

where $\imath$ denotes the imaginary unit;

2. *the eigenvectors of the preconditioned matrix $\boldsymbol{P_{BCD}^{-1}A}$ are*

$$\begin{pmatrix} \mathbf{x}^{(l)} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ -\boldsymbol{M}^{-1}\boldsymbol{K}^T\mathbf{x}^{(l)} \\ \mathbf{0} \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{x}^{(l)} \end{pmatrix}, \quad l = 1, \ldots, m.$$

Theorem 1.2. (Theorem 3.1 in [3]) *Let $\boldsymbol{A} \in \mathbb{R}^{3m\times 3m}$ be the coefficient matrix of the saddle-point problem* (1.4) *and $\boldsymbol{P_{BCT}} \in \mathbb{R}^{3m\times 3m}$ be the block-counter-triangular preconditioner of $\boldsymbol{A}$ defined in* (1.6). *Assume that v_l is an eigenvalue and $\mathbf{x}^{(l)} \in \mathbb{C}^m$ is the corresponding eigenvector of the matrix $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T \in \mathbb{R}^{m\times m}$, $l = 1,\dots,m$, where $v_l > 0$ $(l = 1,\dots,m)$. Then*

1. *the eigenvalues of the preconditioned matrix $\boldsymbol{P}^{-1}_{\boldsymbol{BCT}}\boldsymbol{A}$ are 1 with algebraic multiplicity $2m$, and $2\beta v_l + 1, l = 1,2,\dots,m$;*
2. *the eigenvectors of the preconditioned matrix $\boldsymbol{P}^{-1}_{\boldsymbol{BCT}}\boldsymbol{A}$ are*

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{y} \\ \mathbf{z} \end{pmatrix}, \quad \forall \mathbf{y},\mathbf{z} \in \mathbb{C}^m\backslash\{0\}, \quad \begin{pmatrix} -v_l\mathbf{x}^{(l)} \\ -\boldsymbol{M}^{-1}\boldsymbol{K}^T\mathbf{x}^{(l)} \\ \mathbf{x}^{(l)} \end{pmatrix}, \quad l = 1,\dots,m.$$

Based on Theorems 1.1 and 1.2, the following remarks can be easily obtained.

Remark 1.1. Let $\boldsymbol{M},\boldsymbol{K} \in \mathbb{R}^{m\times m}$ be the block matrices of the saddle-point problem (1.4). Assume that v_l is an eigenvalue and $\mathbf{x}^{(l)} \in \mathbb{C}^m$ is the corresponding eigenvector of the matrix $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T \in \mathbb{R}^{m\times m}$, $l = 1,2,\dots,m$, where $v_l > 0$ $(l = 1,2,\dots,m)$. Let $\boldsymbol{\Lambda} = \mathrm{diag}(v_1,v_2,\cdots,v_m)$, and $\boldsymbol{X} \in \mathbb{C}^{m\times m}$ be an eigenvector matrix of $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$, that is, the lth column of $\boldsymbol{X}$ is $\mathbf{x}^{(l)}$ $(l = 1,2,\cdots,m)$. Then $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$ can be diagonalized as $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}$.

Remark 1.2. Since $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T \in \mathbb{R}^{m\times m}$ is diagonalizable, the preconditioned matrices $\boldsymbol{P}^{-1}_{\boldsymbol{BCD}}\boldsymbol{A}$ and $\boldsymbol{P}^{-1}_{\boldsymbol{BCT}}\boldsymbol{A} \in \mathbb{R}^{3m\times 3m}$ can be diagonalizable, too. And if the eigenvalues of $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$ are clustered or if β is small, then the eigenvalues of $\boldsymbol{P}^{-1}_{\boldsymbol{BCD}}\boldsymbol{A}$ and $\boldsymbol{P}^{-1}_{\boldsymbol{BCT}}\boldsymbol{A}$ are clustered, too.

Remark 1.3. The splittings $\boldsymbol{A} = \boldsymbol{P_{BCD}} - \boldsymbol{R_{BCD}}$ and $\boldsymbol{A} = \boldsymbol{P_{BCT}} - \boldsymbol{R_{BCT}}$ of the saddle-point matrix $\boldsymbol{A} \in \mathbb{R}^{3m\times 3m}$, induced by $\boldsymbol{P_{BCD}}$ and $\boldsymbol{P_{BCT}}$, are convergent if and only if the matrix $2\beta\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$ is convergent. Moreover, it holds that

$$\rho(\boldsymbol{P}^{-1}_{\boldsymbol{BCD}}\boldsymbol{R_{BCD}}) \le \sqrt[3]{2\beta\rho(\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T)},$$
$$\rho(\boldsymbol{P}^{-1}_{\boldsymbol{BCT}}\boldsymbol{R_{BCT}}) \le 2\beta\rho(\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T),$$

with $\rho(\cdot)$ representing the spectral radius of the corresponding matrix.

When we apply the GMRES method to solve a linear system with a nonsingular coefficient matrix $\boldsymbol{B} \in \mathbb{R}^{3m\times 3m}$, it is well known that if $\boldsymbol{B}$ is diagonalizable with its eigenvector matrix $\boldsymbol{E}$, then the 2-norm k-th residual of the GMRES method is bounded from above as

$$\mathrm{RES} = \frac{\|r_k\|_2}{\|r_0\|_2} \le \kappa_2(\boldsymbol{E}) \min_{p\in\mathbb{P}_k, p(0)=1} \max_{\lambda\in\mathbb{E}(\boldsymbol{B})} |\ p_k(\lambda)\ |, \tag{1.7}$$

where $\kappa_2(\boldsymbol{E})$ is the 2-norm condition number of matrix $\boldsymbol{E}$, $\mathbb{P}_k$ is the set of polynomials of degree not greater than k, and $\mathbb{E}(\boldsymbol{B})$ is a set containing the spectra of matrix $\boldsymbol{B}$, the details can be found in [4,14]. The convergence of GMRES is therefore essentially bounded by quantity

$$\rho_k(\mathbb{E}(\boldsymbol{B})) = \min_{p\in\mathbb{P}_k, p(0)=1} \max_{\lambda\in\mathbb{E}(\boldsymbol{B})} |\ p_k(\lambda)\ |\ .$$

The corresponding asymptotic convergence factor (see [15]) is defined by

$$\rho(\mathbb{E}(\boldsymbol{B})) = (\rho_k(\mathbb{E}(\boldsymbol{B})))^{\frac{1}{k}}.$$

In this paper, we focus on the estimations of $\kappa_2(\boldsymbol{E})$ and $\rho(\mathbb{E}(\boldsymbol{B}))$ for the preconditioned matrices $\boldsymbol{B} = \boldsymbol{P}_{\boldsymbol{BCD}}^{-1}\boldsymbol{A}$ and $\boldsymbol{B} = \boldsymbol{P}_{\boldsymbol{BCT}}^{-1}\boldsymbol{A}$, respectively.

$\rho(\mathbb{E}(\boldsymbol{B}))$ can be estimated by utilizing the following corollary.

Corollary 1.1. *Let $\boldsymbol{B}$ be a diagonalizable matrix; i.e., $\boldsymbol{B} = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1}$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the diagonal matrix of the eigenvalues of $\boldsymbol{B}$. Assume that all eigenvalues of $\boldsymbol{B}$ are located in the ellipse $\boldsymbol{\mathcal{E}}(a,b,c,d)$ with center d, foci $d \pm c$ and semi-axis a and b, where $c^2 = a^2 - b^2$. Note that the ellipse $\boldsymbol{\mathcal{E}}(a,b,c,d)$ has either real or complex conjugate foci depending on the sign of $a-b$. Then the asymptotic convergence factor of the GMRES method on this ellipse can be bounded by*

$$\rho(\mathbb{E}(\boldsymbol{B})) = \frac{a+b}{d+\sqrt{d^2-c^2}}.$$

We refer the readers to [4, 14] *for details.*

The outline of this paper is as follows. In Section 2, we give convergence analyses for the preconditioned GMRES methods with the block-counter-diagonal preconditioner $\boldsymbol{P}_{\boldsymbol{BCD}}$ and the block-counter-triangular preconditioner $\boldsymbol{P}_{\boldsymbol{BCT}}$ for solving (1.4), respectively. In Section 3, numerical examples are performed to illustrate the theoretical results. Finally, in Section 4, we use a brief conclusion to end the paper.

2. The Convergence Analysis for the Preconditioned GMRES

We consider the convergence properties of the preconditioned GMRES methods when solving (1.4) with preconditioners $\boldsymbol{P}_{\boldsymbol{BCD}}$ and $\boldsymbol{P}_{\boldsymbol{BCT}}$, respectively.

2.1. Convergence for the Preconditioned GMRES with $\boldsymbol{P}_{\boldsymbol{BCD}}$

Firstly, we focus on analyzing the preconditioned matrix $\boldsymbol{P}_{\boldsymbol{BCD}}^{-1}\boldsymbol{A}$. For simplicity, we denote $\boldsymbol{A}_{\boldsymbol{PBCD}} = \boldsymbol{P}_{\boldsymbol{BCD}}^{-1}\boldsymbol{A}$. From Theorem 1.1 and Remark 1.1, the eigenvector matrix $\boldsymbol{X}_{\boldsymbol{PBCD}}$ of the preconditioned matrix $\boldsymbol{A}_{\boldsymbol{PBCD}}$ can be written as

$$\boldsymbol{X}_{\boldsymbol{PBCD}} = \mathrm{diag}(\boldsymbol{X}, -\boldsymbol{M}^{-1}\boldsymbol{K}^T\boldsymbol{X}, \boldsymbol{X}). \tag{2.1}$$

Remark 1.2 shows that the preconditioned matrix $\boldsymbol{A}_{\boldsymbol{PBCD}}$ is diagonalizable, i.e.,

$$\boldsymbol{A}_{\boldsymbol{PBCD}} = \boldsymbol{X}_{\boldsymbol{PBCD}}\boldsymbol{\Lambda}_{\boldsymbol{PBCD}}\boldsymbol{X}_{\boldsymbol{PBCD}}^{-1}$$

with $\boldsymbol{\Lambda}_{\boldsymbol{PBCD}}$ being a diagonal matrix whose diagonal elements are the eigenvalues of the preconditioned matrix $\boldsymbol{A}_{\boldsymbol{PBCD}}$ defined in Theorem 1.1. The following lemma shows that there exists an eigenvector matrix $\boldsymbol{X}$ such that $\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}$ is orthogonal.

Lemma 2.1. *Let $\boldsymbol{M}, \boldsymbol{K} \in \mathbb{R}^{m\times m}$ be the block matrices of the saddle-point problem* (1.4)*, then $\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T\boldsymbol{M}^{-\frac{1}{2}}$ is similar to $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$. In addition, there exists an eigenvector matrix $\boldsymbol{X}' \in \mathbb{C}^{m\times m}$ of $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$ such that $\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}'$ is orthogonal.*

Proof. Let $\boldsymbol{P}$ be an orthogonal eigenvector matrix of the symmetric matrix $\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T\boldsymbol{M}^{-\frac{1}{2}}$. Since

$$\boldsymbol{M}^{\frac{1}{2}}(\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T)\boldsymbol{M}^{-\frac{1}{2}} = \boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T\boldsymbol{M}^{-\frac{1}{2}},$$

the matrix $\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T\boldsymbol{M}^{-\frac{1}{2}}$ is similar to $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$ by $\boldsymbol{M}^{\frac{1}{2}}$, and $\boldsymbol{X}' = \boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{P}$ is an eigenvector matrix of $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T$, it easily follows that $\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}'$ is an orthogonal matrix. □

For the condition number of the matrix $\boldsymbol{X_{PBCD}}$ with matrix $\boldsymbol{X} = \boldsymbol{X}'$ defined in Lemma 2.1, we have the following theorem.

Theorem 2.1. *Let $\boldsymbol{A} \in \mathbb{R}^{3m\times 3m}$ be the coefficient matrix of the saddle-point problem* (1.4), *$\boldsymbol{P_{BCD}} \in \mathbb{R}^{3m\times 3m}$ be the block-counter-diagonal preconditioner of $\boldsymbol{A}$ defined in* (1.5), *and $\boldsymbol{X_{PBCD}} \in \mathbb{R}^{3m\times 3m}$ be the eigenvector matrix of the preconditioned matrix $\boldsymbol{P^{-1}_{BCD}A}$ defined in* (2.1). *Assume that v_l is an eigenvalue of $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T \in \mathbb{R}^{m\times m}$, $l = 1, 2, \ldots, m$, and $\boldsymbol{\Lambda} = \mathrm{diag}(v_1, v_2, \cdots, v_m)$, where $v_l > 0$. Then*

$$\begin{aligned}\kappa_2(\boldsymbol{X_{PBCD}}) &= \|\boldsymbol{X_{PBCD}}\|_2\|\boldsymbol{X^{-1}_{PBCD}}\|_2 \\ &\le \kappa_2(\boldsymbol{M})\sqrt{\max_{1\le l\le m}\{v_l, 1\}\max_{1\le l\le m}\{v_l^{-1}, 1\}}.\end{aligned} \tag{2.2}$$

Proof. Let

$$\boldsymbol{R} = \mathrm{diag}(\boldsymbol{M}^{\frac{1}{2}}, \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}, \boldsymbol{M}^{\frac{1}{2}}), \quad \boldsymbol{C} = \boldsymbol{R}\boldsymbol{A_{PBCD}}\boldsymbol{R}^{-1}.$$

Then $\boldsymbol{C}$ is similar to $\boldsymbol{A_{PBCD}}$ and

$$\boldsymbol{C} = \boldsymbol{R}\boldsymbol{X_{PBCD}}\boldsymbol{\Lambda_{PBCD}}\boldsymbol{X^{-1}_{PBCD}}\boldsymbol{R}^{-1} = \boldsymbol{Y}\boldsymbol{\Lambda_{PBCD}}\boldsymbol{Y}^{-1},$$

where

$$\boldsymbol{Y} = \boldsymbol{R}\boldsymbol{X_{PBCD}} = \mathrm{diag}(\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}, -\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}, \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}).$$

From Lemma 2.1, $\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}$ is an orthogonal matrix, we immediately know that $\boldsymbol{Y}$ is also orthogonal and $\boldsymbol{X_{PBCD}} = \boldsymbol{R}^{-1}\boldsymbol{Y}$. Therefore,

$$\begin{aligned}\|\boldsymbol{X_{PBCD}}\|_2^2 &= \rho(\boldsymbol{X_{PBCD}}\boldsymbol{X^T_{PBCD}}) = \rho(\boldsymbol{R}^{-1}\boldsymbol{Y}\boldsymbol{Y}^T\boldsymbol{R}^{-T}) = \|\boldsymbol{R}^{-1}\|_2^2, \\ \|\boldsymbol{X^{-1}_{PBCD}}\|_2^2 &= \rho(\boldsymbol{X^{-T}_{PBCD}}\boldsymbol{X^{-1}_{PBCD}}) = \rho(\boldsymbol{R}^T\boldsymbol{Y}^{-T}\boldsymbol{Y}^{-1}\boldsymbol{R}) = \|\boldsymbol{R}\|_2^2.\end{aligned}$$

Consequently,

$$\|\boldsymbol{X_{PBCD}}\|_2\|\boldsymbol{X^{-1}_{PBCD}}\|_2 = \kappa_2(\boldsymbol{R}). \tag{2.3}$$

By direct computations we have

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{W}\boldsymbol{U},$$

where

$$\boldsymbol{W} = \mathrm{diag}(\boldsymbol{M}, \boldsymbol{M}, \boldsymbol{M}), \tag{2.4}$$

and $\boldsymbol{U}$ can be expressed as

$$\boldsymbol{U} = \mathrm{diag}(\boldsymbol{I}, \boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{K}^{-T}\boldsymbol{M}, \boldsymbol{I}) = \mathrm{diag}(\boldsymbol{I}, \boldsymbol{X}\boldsymbol{\Lambda}^{-1}\boldsymbol{X}^{-1}, \boldsymbol{I}), \tag{2.5}$$

based on Remark 1.1, $\boldsymbol{I} \in \mathbb{R}^{m\times m}$ is the identity matrix. As it holds that

$$\|\boldsymbol{W}\|_2 = \|\boldsymbol{M}\|_2, \tag{2.6}$$

we have

$$\|\boldsymbol{R}\|_2^2 = \|\boldsymbol{R}^T\boldsymbol{R}\|_2 = \|\boldsymbol{W}\boldsymbol{U}\|_2 \le \|\boldsymbol{M}\|_2\|\boldsymbol{U}\|_2, \tag{2.7}$$

$$\|\boldsymbol{R}^{-1}\|_2^2 = \|\boldsymbol{R}^{-1}\boldsymbol{R}^{-T}\|_2 = \|\boldsymbol{U}^{-1}\boldsymbol{W}^{-1}\|_2 \le \|\boldsymbol{U}^{-1}\|_2\|\boldsymbol{M}^{-1}\|_2. \tag{2.8}$$

From (2.3), (2.7) and (2.8), it is easy to obtain

$$\|\boldsymbol{X_{PBCD}}\|_2^2\|\boldsymbol{X^{-1}_{PBCD}}\|_2^2 \le \kappa_2(\boldsymbol{M})\kappa_2(\boldsymbol{U}). \tag{2.9}$$

It follows from (2.5) that

$$\begin{aligned}\kappa_2(\boldsymbol{U}) &= \max\{\|\boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1}\|_2, 1\}\max\{\|\boldsymbol{X}\boldsymbol{\Lambda}^{-1}\boldsymbol{X}^{-1}\|_2, 1\}\\ &\le \|\boldsymbol{X}\|_2^2\|\boldsymbol{X}^{-1}\|_2^2 \max_{1\le l\le m}\{v_l, 1\}\max_{1\le l\le m}\{v_l^{-1}, 1\}.\end{aligned}$$

Note that the matrix $\boldsymbol{D} = \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X}$ is orthogonal, and $\boldsymbol{X} = \boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{D}$. So

$$\|\boldsymbol{X}\|_2^2 = \rho(\boldsymbol{M}^{-\frac{1}{2}}\boldsymbol{D}\boldsymbol{D}^T\boldsymbol{M}^{-\frac{1}{2}}) = \rho(\boldsymbol{M}^{-1}) = \|\boldsymbol{M}^{-1}\|_2, \tag{2.10}$$

$$\|\boldsymbol{X}^{-1}\|_2^2 = \rho(\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{D}^{-T}\boldsymbol{D}^{-1}\boldsymbol{M}^{\frac{1}{2}}) = \rho(\boldsymbol{M}) = \|\boldsymbol{M}\|_2. \tag{2.11}$$

We have

$$\kappa_2(\boldsymbol{U}) \le \kappa_2(\boldsymbol{M})\max_{1\le l\le m}\{v_l, 1\}\max_{1\le l\le m}\{v_l^{-1}, 1\}. \tag{2.12}$$

The desired result (2.2) then follows from (2.9) and (2.12). □

Now we consider the asymptotic convergence factor of the GMRES method on the ellipse containing the eigenvalues of the preconditioned matrix $\boldsymbol{P^{-1}_{BCD}A}$. We have shown in Theorem 1.1 that the eigenvalues of the preconditioned matrix are enclosed in the rectangle

$$[1-q,\ 1+q]\times[-q,\ q],$$

where

$$q = \max_{1\le l\le m}|\sqrt[3]{2\beta v_l}|,$$

see also [4]. To estimate the asymptotic convergence rate of the preconditioned GMRES method based on Corollary 1.1, we compute an ellipse $\boldsymbol{\mathcal{E}}(a_1, b_1, c_1, d_1)$ of the estimated smallest area containing this rectangle. Because the center of the rectangle is $(\tau_1, 0)$, where $\tau_1 = 1$, and the lengths of the sides of the rectangle are $\chi_1 = 2q$ and $\omega_1 = 2q$. The ellipse $\mathcal{E}(a_1, b_1, c_1, d_1)$ which has the smallest area of all ellipses and encloses the rectangle is given by

$$a_1 = \frac{\sqrt{2}}{2}\chi_1, \quad b_1 = \frac{\sqrt{2}}{2}\omega_1, \quad c_1 = \frac{\sqrt{2}}{2}\sqrt{|\omega_1^2 - \chi_1^2|}, \quad d_1 = \tau_1.$$

Hence

$$\rho_k(\mathbb{E}(\boldsymbol{P^{-1}_{BCD}A})) \;=\; \frac{\chi_1 + \omega_1}{\sqrt{2}\tau_1 + \sqrt{2\tau_1^2 - |\chi_1^2 - \omega_1^2|}} \;=\; \sqrt{2}q. \tag{2.13}$$

It follows from (2.13) that when

$$0 \le q < \frac{\sqrt{2}}{2} \simeq 0.7071,$$

the asymptotic convergence factor $\rho_k(\mathbb{E}(\boldsymbol{P^{-1}_{BCD}A})) < 1$. By substituting (2.2) and (2.13) into (1.7), we have the k-th 2-norm residual of the preconditioned GMRES with preconditioner $\boldsymbol{P_{BCD}}$ as

$$\mathrm{RES}_{\mathrm{PBCD}} \;=\; \frac{\|r_k\|_2}{\|r_0\|_2} \;\le\; (\sqrt{2}q)^k\kappa_2(\boldsymbol{M})\sqrt{\max_{1\le l\le m}\{v_l, 1\}\max_{1\le l\le m}\{v_l^{-1}, 1\}}.$$

2.2. Convergence for the Preconditioned GMRES with $\boldsymbol{P_{BCT}}$

Now we focus on analyzing the preconditioned matrix $\boldsymbol{P_{BCT}^{-1}A}$. For simplicity, we denote $\boldsymbol{A_{PBCT}} = \boldsymbol{P_{BCT}^{-1}A}$. Theorem 1.2 and Remark 1.2 show that the preconditioned matrix $\boldsymbol{A_{PBCT}}$ can be diagonalized as

$$\boldsymbol{A_{PBCT}} = \boldsymbol{X_{PBCT}\Lambda_{PBCT}X_{PBCT}^{-1}},$$

with $\boldsymbol{\Lambda_{PBCT}}$ being a diagonal matrix whose diagonal elements are eigenvalues of the preconditioned matrix $\boldsymbol{A_{PBCT}}$ defined in Theorem 1.2, and $\boldsymbol{X_{PBCT}}$ is the corresponding eigenvector matrix. By using Remark 1.1, the eigenvector matrix $\boldsymbol{X_{PBCT}}$ of the preconditioned matrix $\boldsymbol{A_{PBCT}}$ can be expressed as

$$\boldsymbol{X_{PBCT}} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T\boldsymbol{X} \\ \boldsymbol{O}_{21} & \boldsymbol{O}_{22} & -\boldsymbol{M}^{-1}\boldsymbol{K}^T\boldsymbol{X} \\ \boldsymbol{O}_{31} & \boldsymbol{O}_{32} & \boldsymbol{X} \end{pmatrix}, \tag{2.14}$$

where $\boldsymbol{X}, \boldsymbol{\Lambda}$ are defined in Remark 1.1, $\boldsymbol{O}_{21}$, $\boldsymbol{O}_{22}$, $\boldsymbol{O}_{31}$ and $\boldsymbol{O}_{32} \in \mathbb{C}^{m\times m}$ are the matrices which are determined in the second part of Theorem 1.2.

For the condition number of the matrix $\boldsymbol{X_{PBCT}}$ with matrix $\boldsymbol{X} = \boldsymbol{X}'$ defined in Lemma 2.1, we have the following theorem.

Theorem 2.2. *Let $\boldsymbol{A} \in \mathbb{R}^{3m\times 3m}$ be the coefficient matrix of the saddle-point problem* (1.4) *and $\boldsymbol{P_{BCT}} \in \mathbb{R}^{3m\times 3m}$ be the block-counter-triangular preconditioner of $\boldsymbol{A}$ defined in* (1.6), *$\boldsymbol{X_{PBCT}} \in \mathbb{R}^{3m\times 3m}$ be the eigenvector matrix of the preconditioned matrix $\boldsymbol{P_{BCT}^{-1}A}$ defined in* (2.14). *Assume that v_l is an eigenvalue of $\boldsymbol{M}^{-1}\boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{K}^T \in \mathbb{R}^{m\times m}$, $l = 1,\ldots,m$, and $\boldsymbol{\Lambda} = \mathrm{diag}(v_1, v_2, \cdots, v_m)$, where $v_l > 0$. Then*

$$\begin{aligned} \kappa_2(\boldsymbol{X_{PBCT}}) &= \|\boldsymbol{X_{PBCT}}\|_2\|\boldsymbol{X_{PBCT}^{-1}}\|_2 \\ &\le (2+\sqrt{3})\kappa_2(\boldsymbol{M})\sqrt{\max_{1\le l\le m}\{v_l^2, v_l, 1\}\max_{1\le l\le m}\{v_l^{-2}, v_l^{-1}, 1\}}. \end{aligned} \tag{2.15}$$

Proof. Let

$$\boldsymbol{L} = \mathrm{diag}(\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M}, \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}, \boldsymbol{M}^{\frac{1}{2}}).$$

By defining

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{I} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}$$

with $\boldsymbol{I} \in \mathbb{R}^{m\times m}$ being the identity matrix, we let

$$\boldsymbol{S} = \boldsymbol{Q}\boldsymbol{L} = \begin{pmatrix} \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M} & \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M} & \boldsymbol{0} & \boldsymbol{M}^{\frac{1}{2}} \end{pmatrix} \tag{2.16}$$

and

$$\boldsymbol{F} = \boldsymbol{S}\boldsymbol{A_{PBCT}}\boldsymbol{S}^{-1}.$$

$\boldsymbol{F}$ is similar to $\boldsymbol{A_{PBCT}}$ and can be written as

$$\boldsymbol{F} = \boldsymbol{S}\boldsymbol{X_{PBCT}\Lambda_{PBCT}X_{PBCT}^{-1}}\boldsymbol{S}^{-1} = \boldsymbol{Z\Lambda_{PBCT}Z}^{-1},$$

where

$$\boldsymbol{Z} = \boldsymbol{S}\boldsymbol{X}_{\boldsymbol{PBCT}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & -\boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X} \\ \bar{\boldsymbol{O}}_{21} & \bar{\boldsymbol{O}}_{22} & \mathbf{0} \\ \bar{\boldsymbol{O}}_{31} & \bar{\boldsymbol{O}}_{32} & \mathbf{0} \end{pmatrix}, \tag{2.17}$$

$\boldsymbol{X}$ is defined in Remark 1.1. Hence, we have

$$\begin{aligned} \bar{\boldsymbol{O}}_{21} &= \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}\boldsymbol{O}_{21}, & \bar{\boldsymbol{O}}_{22} &= \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{K}^{-T}\boldsymbol{M}\boldsymbol{O}_{22}, \\ \bar{\boldsymbol{O}}_{31} &= \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{O}_{31}, & \bar{\boldsymbol{O}}_{32} &= \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{O}_{32}. \end{aligned}$$

Under the restrictions of Theorem 1.2 for (2.14), we select matrices

$$\boldsymbol{O}_{21} = \boldsymbol{M}^{-1}\boldsymbol{K}^{T}\boldsymbol{X}, \quad \boldsymbol{O}_{22} = \mathbf{0}, \quad \boldsymbol{O}_{31} = \mathbf{0}, \quad \boldsymbol{O}_{32} = \boldsymbol{X}.$$

Then

$$\begin{pmatrix} \bar{\boldsymbol{O}}_{21} & \bar{\boldsymbol{O}}_{22} \\ \bar{\boldsymbol{O}}_{31} & \bar{\boldsymbol{O}}_{32} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}^{\frac{1}{2}}\boldsymbol{X} \end{pmatrix}. \tag{2.18}$$

By substituting (2.18) into (2.17), we easily know that the matrix $\boldsymbol{Z}$ is orthogonal. Since $\boldsymbol{X}_{\boldsymbol{PBCT}} = \boldsymbol{S}^{-1}\boldsymbol{Z}$, we have

$$\|\boldsymbol{X}_{\boldsymbol{PBCT}}\|_2^2 = \rho(\boldsymbol{X}_{\boldsymbol{PBCT}}\boldsymbol{X}_{\boldsymbol{PBCT}}{}^T) = \rho(\boldsymbol{S}^{-1}\boldsymbol{S}^{-T}) = \|\boldsymbol{S}^{-1}\|_2^2, \tag{2.19}$$

$$\|\boldsymbol{X}_{\boldsymbol{PBCT}}^{\boldsymbol{-1}}\|_2^2 = \rho(\boldsymbol{X}_{\boldsymbol{PBCT}}^{\boldsymbol{-1}}(\boldsymbol{X}_{\boldsymbol{PBCT}}^{\boldsymbol{-1}})^T) = \rho(\boldsymbol{S}\boldsymbol{S}^T) = \|\boldsymbol{S}\|_2^2. \tag{2.20}$$

It follows from (2.16), (2.19) and (2.20) that

$$\|\boldsymbol{X}_{\boldsymbol{PBCT}}\|_2\|\boldsymbol{X}_{\boldsymbol{PBCT}}^{\boldsymbol{-1}}\|_2 = \|\boldsymbol{S}^{-1}\|_2\|\boldsymbol{S}\|_2 \le \kappa_2(\boldsymbol{Q})\kappa_2(\boldsymbol{L}). \tag{2.21}$$

It follows from straightforward computations that

$$\boldsymbol{L}^T\boldsymbol{L} = \boldsymbol{W}\boldsymbol{V},$$

where $\boldsymbol{W}$ is defined in (2.4) and

$$\boldsymbol{V} = \mathrm{diag}((\boldsymbol{K}^{-T}\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M})^2, \boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{K}^{-T}\boldsymbol{M}, \boldsymbol{I}).$$

From (2.6), we obtain

$$\begin{aligned} \|\boldsymbol{L}\|_2^2 &= \|\boldsymbol{L}^T\boldsymbol{L}\|_2 = \|\boldsymbol{W}\boldsymbol{V}\|_2 \le \|\boldsymbol{M}\|_2\|\boldsymbol{V}\|_2, \\ \|\boldsymbol{L}^{-1}\|_2^2 &= \|\boldsymbol{L}^{-1}\boldsymbol{L}^{-T}\|_2 = \|\boldsymbol{V}^{-1}\boldsymbol{W}^{-1}\|_2 \le \|\boldsymbol{M}^{-1}\|_2\|\boldsymbol{V}^{-1}\|_2. \end{aligned}$$

Consequently,

$$\kappa_2(\boldsymbol{L}) \le \sqrt{\kappa_2(\boldsymbol{M})}\sqrt{\kappa_2(\boldsymbol{V})}. \tag{2.22}$$

Based on Remark 1.1, we know that

$$\boldsymbol{V} = \bar{\boldsymbol{X}}\boldsymbol{N}\bar{\boldsymbol{X}}^{\boldsymbol{-1}},$$

where

$$\bar{\boldsymbol{X}} = \mathrm{diag}(\boldsymbol{X}, \boldsymbol{X}, \boldsymbol{X}) \quad \text{and} \quad \boldsymbol{N} = \mathrm{diag}(\boldsymbol{\Lambda}^{-2}, \boldsymbol{\Lambda}^{\boldsymbol{-1}}, \boldsymbol{I}).$$

Hence,

$$\boldsymbol{V^{-1}} = \boldsymbol{\bar{X} N^{-1} \bar{X}^{-1}}.$$

By using (2.10) and (2.11), we have

$$\begin{aligned}
\kappa_2(\boldsymbol{V}) &= \|\boldsymbol{\bar{X} N \bar{X}^{-1}}\|_2 \|\boldsymbol{\bar{X} N^{-1} \bar{X}^{-1}}\|_2 \\
&\leq \|\boldsymbol{X}\|_2^2 \|\boldsymbol{X^{-1}}\|_2^2 \|\boldsymbol{N}\|_2 \|\boldsymbol{N^{-1}}\|_2 = \kappa_2(\boldsymbol{M}) \|\boldsymbol{N}\|_2 \|\boldsymbol{N^{-1}}\|_2 \\
&\leq \kappa_2(\boldsymbol{M}) \max\{\|\boldsymbol{\Lambda}^{-2}\|_2, \|\boldsymbol{\Lambda}^{-1}\|_2, 1\} \ \max\{\|\boldsymbol{\Lambda}^{2}\|_2, \|\boldsymbol{\Lambda}\|_2, 1\} \\
&= \kappa_2(\boldsymbol{M}) \max_{1\leq l\leq m}\{v_l^2, v_l, 1\} \max_{1\leq l\leq m}\{v_l^{-2}, v_l^{-1}, 1\}. \qquad (2.23)
\end{aligned}$$

Through direct calculations, we get $\kappa_2(\boldsymbol{Q}) = 2+\sqrt{3}$. Finally from (2.21), (2.22) and (2.23), we obtain (2.15). □

Now we consider the asymptotic convergence factor of the GMRES method on the ellipse containing the eigenvalues of the preconditioned matrix $\boldsymbol{P_{BCT}^{-1}A}$. We have shown in Theorem 1.2 that the eigenvalues of the preconditioned matrix are enclosed in the rectangle

$$[1,\ 1+p] \times [0,\ 0],$$

where

$$p = \max_{1\leq l\leq m} 2\beta v_l,$$

see, for instance, [4]. To estimate the asymptotic convergence rate of the preconditioned GMRES method based on Corollary 1.1, we compute an ellipse $\boldsymbol{\mathcal{E}}(a_2, b_2, c_2, d_2)$ of the smallest area containing this rectangle. Because the center of the rectangle is $(\tau_2, 0)$, where $\tau_2 = 1 + 0.5\ p$, and the lengths of the sides of the rectangle are $\chi_2 = p$ and $\omega_2 = 0$, the ellipse $\mathcal{E}(a_2, b_2, c_2, d_2)$ which has the estimated smallest area of all ellipses and encloses the rectangle is given by

$$a_2 = \frac{\sqrt{2}}{2}\chi_2, \quad b_2 = \frac{\sqrt{2}}{2}\omega_2, \quad c_2 = \frac{\sqrt{2}}{2}\sqrt{|\omega_2^2 - \chi_2^2|}, \quad d_2 = \tau_2.$$

Hence,

$$\begin{aligned}
\rho_k(\mathbb{E}(\boldsymbol{P_{BCT}^{-1}A})) &= \frac{\chi_2 + \omega_2}{\sqrt{2}\tau_2 + \sqrt{2\tau^2 - |\chi^2 - \omega^2|}} \\
&= \frac{p}{\sqrt{2}(1+0.5p) + \sqrt{2+2p-0.5p^2}}. \qquad (2.24)
\end{aligned}$$

It follows from (2.24) that when

$$0 \leq p \leq \frac{\sqrt{2} + \sqrt{2+16\sqrt{2}}}{4} \simeq 1.5942,$$

the asymptotic convergence factor $\rho_k(\mathbb{E}(\boldsymbol{P_{BCT}^{-1}A})) < 1$. By substituting (2.15) and (2.24) into (1.7), we have the k-th 2-norm residual of the preconditioned GMRES with preconditioner $\boldsymbol{P_{BCT}}$ as

$$\mathrm{RES_{PBCT}} \leq \frac{(2+\sqrt{3})p^k \kappa_2(\boldsymbol{M}) \sqrt{\max_{1\leq l\leq m}\{v_l^2, v_l, 1\} \ \max_{1\leq l\leq m}\{v_l^{-2}, v_l^{-1}, 1\}}}{(\sqrt{2}(1+0.5p) + \sqrt{2+2p-0.5p^2})^k}.$$

3. Numerical Results

In this section, computations are performed for the example adopted from [3] to compare the convergence behaviors obtained from the numerical outcomes and the theoretical results shown in the previous sections. The experiments are run in MATLAB(version 7.8) with a machine precision 10^{-16}.

Example 3.1. [3] *Let* $\Omega = [0,1]^2$ *be a unit square and consider the distributed control problem* (1.1)-(1.3), *with* $\partial\Omega_2 = \emptyset$, $\mathbf{g} = \mathbf{u}_*$ *and*

$$\mathbf{u}_* = \begin{cases} (2x-1)^2(2y-1)^2, & \text{if } \ (\mathrm{x,y}) \in [0, \frac{1}{2}]^2, \\ 0, & \text{otherwise.} \end{cases}$$

In [3], the system of linear equations (1.4) resulting from Example 3.1 is solved by using preconditioned GMRES with preconditioners $\boldsymbol{P_{BCD}}$ and $\boldsymbol{P_{BCT}}$, respectively. The computing results show that when the proposed preconditioners are used in GMRES for solving (1.4), the iteration steps and computing times are greatly reduced. In addition, the numbers of iteration steps are almost independent of the mesh step-size h when the value of β is small.

For the sake of simplicity, we define the estimated upper bounds for $\kappa_2(\boldsymbol{X_{PBCD}})$ in Theorem 2.1 and $\kappa_2(\boldsymbol{X_{PBCT}})$ in Theorem 2.2 as

$$\Delta_1 = \kappa_2(\boldsymbol{M})\sqrt{\max_{1\le l\le m}\{v_l, 1\} \max_{1\le l\le m}\{v_l^{-1}, 1\}},$$

$$\Delta_2 = (2+\sqrt{3})\kappa_2(\boldsymbol{M}) \sqrt{\max_{1\le l\le m}\{v_l^2, v_l, 1\} \max_{1\le l\le m}\{v_l^{-2}, v_l^{-1}, 1\}},$$

respectively. Note that the estimated uppers bounds Δ_1 and Δ_2 are dependent on the mesh step-size h and independent of β. Tests for different step-size $h = 2^{-3}, 2^{-4}, 2^{-5}$ are completed in our experiments. We show the estimated bounds Δ_1 of $\kappa_2(\boldsymbol{X_{PBCD}})$ in the first row of Table 3.1, and the conditioner numbers $\kappa_2(\boldsymbol{X_{PBCD}})$ are shown in the second row of this table. The third row shows the ratio between Δ_1 and $\kappa_2(\boldsymbol{X_{PBCD}})$ given by

$$r_1 = \frac{\Delta_1}{\kappa_2(\boldsymbol{X_{PBCD}})}.$$

Table 3.1: Δ_1 and $\kappa_2(\boldsymbol{X_{PBCD}})$ for the preconditioned matrix $\boldsymbol{P_{BCD}^{-1}A}$.

h	2^{-3}	2^{-4}	2^{-5}
Δ_1	1.0136×10^4	5.1065×10^4	2.1680×10^5
$\kappa_2(\boldsymbol{X_{PBCD}})$	3.5472×10^3	2.3125×10^4	1.8704×10^5
r_1	2.8574	2.2082	1.1591

Table 3.2: $\rho(\mathbb{E}(\boldsymbol{P_{BCD}^{-1}A}))$ for the preconditioned GMRES.

h	2^{-3}	2^{-4}	2^{-5}
$\beta = 10^{-10}$	1.0217×10^{-1}	2.7218×10^{-1}	6.9573×10^{-1}
$\beta = 10^{-12}$	2.2011×10^{-2}	5.8639×10^{-2}	1.4989×10^{-1}

Table 3.1 shows that the values of Δ_1 become closer to $\kappa_2(\boldsymbol{X_{PBCD}})$ when h decreases. If $h = 2^{-3}$ and $h = 2^{-4}$, the ratio r_1 is greater than 2. But for $h = 2^{-5}$, r_1 is just 1.1591, which is

close to 1. So Δ_1 is close to $\kappa_2(\boldsymbol{X_{PBCD}})$. So the estimated value of $\kappa_2(\boldsymbol{X_{PBCD}})$ in Theorem 2.1 is a good upper bound for $\kappa_2(\boldsymbol{X_{PBCD}})$. The asymptotic convergence factors (2.13) for the preconditioned GMRES method with $\boldsymbol{P_{BCD}}$ for $\beta = 10^{-10}, 10^{-12}$ and $h = 2^{-3}, 2^{-4}, 2^{-5}$ are calculated and shown in Table 3.2. We find that the factors are small for very small values of β, which demonstrates that the preconditioned GMRES method proposed in [3] is very efficient for solving (1.4) especially when the value of β is very small.

Table 3.3: Δ_2 and $\kappa_2(\boldsymbol{X_{PBCT}})$ for the preconditioned matrix $\boldsymbol{P_{BCT}^{-1}A}$.

h	2^{-3}	2^{-4}	2^{-5}
Δ_2	5.1940×10^7	1.1378×10^9	1.9741×10^{10}
$\kappa_2(\boldsymbol{X_{PBCT}})$	4.8704×10^6	1.3806×10^8	3.8203×10^9
r_2	10.6644	8.2413	5.1674

Table 3.4: $\rho(\mathbb{E}(\boldsymbol{P_{BCT}^{-1}A}))$ for the preconditioned GMRES.

h	2^{-3}	2^{-4}	2^{-5}
$\beta = 10^{-10}$	1.333×10^{-4}	2.5115×10^{-3}	3.9794×10^{-2}
$\beta = 10^{-12}$	1.333×10^{-6}	2.5203×10^{-5}	4.2071×10^{-4}

For the preconditioned matrix $\boldsymbol{P_{BCT}^{-1}A}$, the estimated upper bounds of $\kappa_2(\boldsymbol{X_{PBCT}})$ are calculated and shown in the first row of Table 3.3, the conditioner numbers $\kappa_2(\boldsymbol{X_{PBCT}})$ are shown in the second row of this table. And the third row shows the ratio between Δ_2 and $\kappa_2(\boldsymbol{X_{PBCT}})$ given by

$$r_2 = \frac{\Delta_2}{\kappa_2(\boldsymbol{X_{PBCT}})}.$$

Table 3.3 also shows that the values of Δ_2 become closer to $\kappa_2(\boldsymbol{X_{PBCT}})$ when h decreases. However, the approximation between Δ_2 and $\kappa_2(\boldsymbol{X_{PBCT}})$ in this table is not as good as the approximation between Δ_1 and $\kappa_2(\boldsymbol{X_{PBCD}})$ in Table 3.1. For example, when $h = 2^{-5}$, the ratio $r_2 = 5.1674$, which is greater than $r_1 = 1.1591$. Hence, the estimation for $\kappa_2(\boldsymbol{X_{PBCT}})$ in Theorem 2.2 still need to be improved. On the other hand, the values of Δ_2 and $\kappa_2(\boldsymbol{X_{PBCT}})$ are about 10^4 times greater than the corresponding values of Δ_1 and $\kappa_2(\boldsymbol{X}_{PBCD})$, so, r_2 shown in Table 3.4 could also be considered as a suitable measure for judging the proximity between large Δ_2 and $\kappa_2(\boldsymbol{X_{PBCT}})$. The asymptotic convergence factors (2.24) for the preconditioned GMRES method with $\boldsymbol{P_{BCT}}$ for $\beta = 10^{-10}, 10^{-12}$ and $h = 2^{-3}, 2^{-4}, 2^{-5}$ are calculated and shown in Table 3.4. We also find that the factors are very small for small values of β, which further demonstrates that the proposed preconditioned GMRES methods are very efficient for solving (1.4) especially when β is very small.

4. Conclusion

We have considered convergence properties of the preconditioned GMRES methods using block-counter-diagonal $\boldsymbol{P_{BCD}}$ and block-counter-triangular $\boldsymbol{P_{BCT}}$ preconditioners proposed in [3] for solving the elliptic PDE-constrained optimization problems. Condition number of an eigenvector matrix which can diagonalize the preconditioned matrix is estimated and an upper bound is derived for $\boldsymbol{P_{BCD}^{-1}A}$ and $\boldsymbol{P_{BCT}^{-1}A}$, respectively. Experimental results have shown that the theoretically estimated bounds can indicate the real condition numbers of the eigenvector matrices. In future work, we will focus on improving the upper bound of $\kappa_2(\boldsymbol{X_{PBCT}})$ to obtain its more accurate estimates.

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