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DIRECTIONAL DO-NOTHING CONDITION FOR THE NAVIER-STOKES EQUATIONS*

Malte Braack
Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4, D-24098 Kiel, Germany
Email: braack@math.uni-kiel.de
Piotr Boguslaw Mucha
Institute of Applied Mathematics and Mechanics, University of Warsaw, ul Banacha 2, 02-097 Warszawa, Poland
Email: p.mucha@mimuw.edu.pl

Abstract

The numerical solution of flow problems usually requires bounded domains although the physical problem may take place in an unbounded or substantially larger domain. In this case, artificial boundaries are necessary. A well established artificial boundary condition for the Navier-Stokes equations discretized by finite elements is the "do-nothing" condition. The reason for this is the fact that this condition appears automatically in the variational formulation after partial integration of the viscous term and the pressure gradient. This condition is one of the most established outflow conditions for Navier-Stokes but there are very few analytical insight into this boundary condition. We address the question of existence and stability of weak solutions for the Navier-Stokes equations with a "directional do-nothing" condition. In contrast to the usual "do-nothing" condition this boundary condition has enhanced stability properties. In the case of pure outflow, the condition is equivalent to the original one, whereas in the case of inflow a dissipative effect appears. We show existence of weak solutions and illustrate the effect of this boundary condition by computation of steady and non-steady flows.

Mathematics subject classification: 35M12, 65M60, 76D03, 76D05.
Key words: Boundary conditions, Navier-Stokes, Outflow condition, Existence.

1. Introduction

The numerical solution of the Navier-Stokes equations usually requires bounded domains even though the physical problem may take place in an unbounded or substantially larger domain. Therefore, it is necessary to establish artificial boundaries. In fluid dynamics, such artificial boundaries often have an "outflow character". The most established outflow boundary condition for finite element discretization of the Navier-Stokes equations is the so-called "do-nothing" condition. That is because this condition appears automatically due to partial integration of the viscous term and the pressure gradient. If no further boundary integral is added to the variational formulation, the "do-nothing" condition is automatically built in. This formulation was already used by Glowinski [9] and by Gresho [8]. Since then, this condition became the most established outflow condition for Navier-Stokes.

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However, there is hardly any analytical insight into the do-nothing boundary condition. The mere question of whether steady weak solutions for Navier-Stokes equations exist together with this do-nothing condition remains unsolved even in 2D. One of the few articles on this topic addresses different variational formulations and its relation to strong formulations with corresponding outflow conditions [10]. Again, the existence of weak solutions for large data remains a vital question for this boundary condition.

In this work, we address the existential question regarding solutions for the Navier-Stokes equations in combination with a "directional do-nothing" (DDN) condition which leads, on one hand, to enhanced stability. On the other hand, however, when looking at the pure outflow, this particular boundary condition is equivalent to a classical do-nothing condition. The difference to the classical do-nothing condition (CDN) is a nonlinear correction, leading to enhanced stability. This implies that there is no need for any smallness condition on the data to ensure existence. We prove in particular the existence of weak solutions and stability in the H^1-norm plus additional boundary control. This additionally implies uniqueness of small solutions. We analyze the steady case and draw a sketch of proof for the evolutionary system.

In [4], Bruneau and Fabrie presented an entire class of alternative outflow conditions involving several parameters. The DDN condition analyzed in this paper, is obtained for a particular choice of those parameters. A few years later, some investigation has been done by Neustupa and Feistauer in [15, 16]. Although this condition has fundamental advantages compared to the CDN condition, this boundary condition is not very popular yet. Our paper is aimed at stimulating the application of this interesting relation by concisely proving its existence, as well as presenting some new analytical and numerical arguments as to why the nonlinear correction can be useful for Navier-Stokes equations.

As usual for Navier-Stokes, uniqueness of obtained steady solutions for large data can not be addressed here. The theory [7,19] delivers some results about un-uniqueness and uniqueness properties. Hence, the answer concerning the uniqueness in our case is highly nontrivial. It may depend on the particular forcing and the geometry of the domain.

In the computational part of this work, we illustrate the quality of the "directional do-nothing" condition for steady and for non-steady flows and compare it with the CDN condition. As a result, we will see that the modified outflow condition reproduces the solution of the standard outflow condition in the case of pure outflow, and it exhibits too much inflow. This is, of course, a favorable property for an outflow condition.

The outline of the paper is as follows: In Section 2 we present the two types of outflow conditions CDN and DDN, and show why it is impossible to answer the question of existence regarding solutions for the CDN condition. In Section 3 we show stability and existence of solutions for the DDN condition. The time-dependent case is treated in Section 4. Finally, we show in Section 5 the effect of the DDN by means of numerical examples and illustrate the differences to the CDN condition.

2. Outflow Conditions for Navier-Stokes

We consider the stationary incompressible Navier-Stokes equation in the domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$,

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \operatorname{div} \mathbb{T}(\boldsymbol{v}, p) = \boldsymbol{f} \quad \text{in } \Omega\,, \tag{2.1}$$

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \Omega\,. \tag{2.2}$$

Here, $p : \Omega \to \mathbb{R}$ denotes the pressure and $\boldsymbol{v} : \Omega \to \mathbb{R}^d$ the velocity field. The usual variational spaces for these functions are $L^2(\Omega)$ for p and the Sobolev space $H^1(\Omega)^d$ for $\boldsymbol{v}$. $\mathbb{T}$ is the asymmetric stress tensor given by

$$\mathbb{T}(\boldsymbol{v}, p) := \nu \nabla \boldsymbol{v} - pI ,$$

where $\nu > 0$ is the constant viscosity coefficient and I denotes the identity matrix. The right hand side $\boldsymbol{f}$ is assumed to be in $L^2(\Omega)^d$. The $L^2(\Omega)$-norm is denoted by $\| \cdot \|$ and the corresponding scalar product by $(\cdot, \cdot)$.

The boundary $\partial\Omega$ is split in a Dirichlet part S_0 and a natural outflow part S_1, $\partial\Omega = S_0 \cup S_1$. We are interested only in the case that S_0 and S_1 are regular. For the proof of existence of solution we will assume that S_1 is uniform C^1-regular. On S_0 we employ homogeneous Dirichlet conditions

$$\boldsymbol{v} = 0 \quad \text{on } S_0 . \tag{2.3}$$

2.1. Classical do-nothing condition (CDN)

We start with the classical approach on S_1 in form of a mixed condition for velocity and pressure,

$$\mathbb{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} = 0 \quad \text{on } S_1 , \tag{2.4}$$

where $\boldsymbol{n}$ denotes the outer normal vector. We shall emphasize that, due to needs of the "do-nothing" condition (2.4), $\mathbb{T}$ is defined by the full velocity gradient $\nabla \boldsymbol{v}$, not just its symmetric part as it is considered usually in physical reasonable models like in [14,18]. For these boundary conditions the variational spaces consist of the L^2-integrable functions in Ω for p and $H^1(\Omega)$ with vanishing divergence and vanishing traces on S_0 for the velocities:

$$\begin{aligned} &Q := L_2(\Omega) , \\ &\boldsymbol{V} := \left\{ \boldsymbol{v} \in H^1(\Omega)^d \, : \, \operatorname{div} \boldsymbol{v} = 0 , \ \boldsymbol{v}|_{S_0} = 0 \text{ a.e.} \right\} . \end{aligned}$$

For right hand sides in the dual, i.e. $\boldsymbol{f} \in \boldsymbol{V}'$, the action of $\boldsymbol{f}$ onto $\boldsymbol{\varphi} \in \boldsymbol{V}$ is denoted by $\langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle$, and its $\boldsymbol{V}'$-norm is simply denoted by $\|\boldsymbol{f}\|_{-1}$, because it is basically the $H^{-1}(\Omega)$-norm. It is easy to verify that the Poiseuille flow in a rectangular domain fulfills the natural outflow condition (2.4). Furthermore, this boundary condition is somehow natural for finite element discretizations, because it is based on the variational formulation in the function spaces. Multiplying (2.1) by test functions, integration over Ω and integration by parts yields for $\boldsymbol{\varphi} \in \boldsymbol{V}$:

$$\begin{aligned} -(\operatorname{div} \mathbb{T}(\boldsymbol{v}, p), \boldsymbol{\varphi}) &= -\nu(\operatorname{div} \nabla \boldsymbol{v}, \boldsymbol{\varphi}) + (\nabla p, \boldsymbol{\varphi}) \\ &= \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - (p, \operatorname{div} \boldsymbol{\varphi}) + \int_{\partial\Omega} (-\nu \nabla \boldsymbol{v} + p) \cdot \boldsymbol{n} \boldsymbol{\varphi} \, ds \\ &= \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - \int_{\partial\Omega} \mathbb{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} \boldsymbol{\varphi} \, ds . \end{aligned}$$

Hence, we obtain the identity

$$-(\operatorname{div} \mathbb{T}(\boldsymbol{v}, p), \boldsymbol{\varphi}) = \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - \int_{S_1} \mathbb{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} \boldsymbol{\varphi} \, ds .$$

Therefore, the weak formulation of (2.1) with the boundary conditions (2.4) becomes simply:

$$\boldsymbol{v} \in \boldsymbol{V}: \quad \big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{\varphi}\big) + \nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{V}\,. \tag{2.5}$$

And here we meet the first obstacle. The formulation (2.5) shall give us a possibility to obtain an information concerning the sought solution. If we test the equation by a solution, i.e. we put $\boldsymbol{\varphi} = \boldsymbol{v}$ in (2.5), and use

$$\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{v}\big) = \frac{1}{2}\int_{S_1} (\boldsymbol{v}\cdot\boldsymbol{n})|\boldsymbol{v}|^2 d\sigma\,, \tag{2.6}$$

we obtain

$$\frac{1}{2}\int_{S_1} (\boldsymbol{v}\cdot\boldsymbol{n})|\boldsymbol{v}|^2 d\sigma + \nu\|\nabla\boldsymbol{v}\|^2 = \langle \boldsymbol{f}, \boldsymbol{v}\rangle\,.$$

Hence,

$$\nu\|\nabla\boldsymbol{v}\|^2 \le \|\boldsymbol{f}\|_{-1}\|\nabla v\| - \frac{1}{2}\int_{S_1} (\boldsymbol{v}\cdot\boldsymbol{n})|\boldsymbol{v}|^2 d\sigma\,. \tag{2.7}$$

Here, the boundary integral in the r.h.s. is, in general, not positive, and we are not able to control the negative part of it. The bound (2.7) in combination with the continuos embedding $H^1(\Omega) \to L^4(S_1)$ by the trace theorem leads us (with constant $c = c(S_1, \Omega)$) to the estimate

$$\nu\|\nabla\boldsymbol{v}\| \le \|\boldsymbol{f}\|_{-1} + \frac{c}{2}\|\nabla\boldsymbol{v}\| \left(\int_{S_1} \Big((\boldsymbol{v}\cdot\boldsymbol{n})_-\Big)^2 d\sigma\right)^{1/2},$$

where

$$(\boldsymbol{v}\cdot\boldsymbol{n})_- := \begin{cases} 0 & \text{for} \quad \boldsymbol{v}\cdot\boldsymbol{n} \ge 0\,, \\ \boldsymbol{v}\cdot\boldsymbol{n} & \text{for} \quad \boldsymbol{v}\cdot\boldsymbol{n} < 0\,. \end{cases}$$

Now, under the following smallness assumption of the inflow across S_1,

$$\left|\int_{S_1} ((\boldsymbol{v}\cdot\boldsymbol{n})_-)^2 d\sigma\right|^{1/2} \le \frac{\nu}{c}\,, \tag{2.8}$$

we obtain

$$\|\nabla\boldsymbol{v}\| \le \frac{2}{\nu}\|\boldsymbol{f}\|_{-1}\,.$$

In summary, we get a bound of the solution $\boldsymbol{v}$ under the assumption (2.8). However, in the general case, without controlling $(\boldsymbol{v}\cdot\boldsymbol{n})_-$ on S_1, there is no chance to obtain such a bound because the nonlinearity in (2.7) is of degree three and does not provide a sign. This is the reason why, without a smallness assumption of type (2.8), not even existence of weak solutions can be proven for the Navier-Stokes equations with this do-nothing condition (CDN).

2.2. Directional do-nothing condition (DDN)

The observation above leads us to the idea to subtract the undesired boundary integral in the variational formulation. We want to keep fine properties of the form (2.4), but also to have the possibility to prove the existence of weak solutions for the steady case with large data. More specifically, we propose the following equation: Find $\boldsymbol{v} \in \boldsymbol{V}$ such that

$$\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{\varphi}\big)+\nu(\nabla\boldsymbol{v},\nabla\boldsymbol{\varphi})-\frac{1}{2}\int_{S_1}(\boldsymbol{v}\cdot\boldsymbol{n})_-\boldsymbol{v}\boldsymbol{\varphi}\,d\sigma=\langle\boldsymbol{f},\boldsymbol{\varphi}\rangle \qquad \forall\boldsymbol{\varphi}\in\boldsymbol{V}. \tag{2.9}$$

The additional boundary integral in (2.9) is the weak formulation of a boundary condition. We will show that the corresponding strong form of the boundary condition becomes

$$\mathbb{T}(\boldsymbol{v},p)\cdot\boldsymbol{n}-\frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{n})_-\boldsymbol{v}=0 \quad \text{at} \quad S_1. \tag{2.10}$$

Due to the directional preference of the condition (2.10), we call it the *directional-do-nothing* condition (DDN). We make the following observations:

- In the case of pure outflow on this boundary, $\boldsymbol{v}\cdot\boldsymbol{n}\geq 0$, the condition (2.10) is just the CDN condition, CDN=DDN. In particular, Poiseuille flow satisfies the directional-do-nothing condition (2.10).
- The condition is compatible with the Stokes case and with the transport term, which is crucial for the convection dominated case.
- We will prove that it guarantees construction of the energy estimate for large data in the stationary and for the transient case.

This work is structured as follows. In the following section we concentrate on the existence of weak solutions for the steady Navier-Stokes equations with the DDN condition. This is done by proving an energy estimate without any smallness assumption. Furthermore, we will show that under enough regularity, the weak solution fulfills the strong form of the DDN condition, i.e. (2.10). In section 4, we treat the evolutionary case. Numerical comparisons are given in the last section.

3. Existence of Solutions

Our goals are the issue of existence for the steady system (2.9) and numerical analysis of solutions in comparison to the results for the CDN condition. The outflow part of the flux $\boldsymbol{v}\cdot\boldsymbol{n}$ will be denoted by

$$(\boldsymbol{v}\cdot\boldsymbol{n})_+ := \boldsymbol{v}\cdot\boldsymbol{n}-(\boldsymbol{v}\cdot\boldsymbol{n})_-\,.$$

Due to the fact that

$$\int_{S_1}(\boldsymbol{v}\cdot\boldsymbol{n})_+|\boldsymbol{v}|^2 d\sigma \geq 0\,,$$

the expression

$$|\!|\!|\boldsymbol{v}|\!|\!| := \left(\frac{1}{2}\int_{S_1}(\boldsymbol{v}\cdot\boldsymbol{n})_+|\boldsymbol{v}|^2 d\sigma+\nu\|\nabla\boldsymbol{v}\|^2\right)^{1/2}$$

is a non-negative form on $\boldsymbol{V}$ which is stronger that the H^1-seminorm. In particular holds for $S_0\neq\emptyset$:

$$\boldsymbol{v}\in\boldsymbol{V},\ |\!|\!|\boldsymbol{v}|\!|\!|=0 \quad\Rightarrow\quad \boldsymbol{v}=0\,.$$

Lemma 3.1. *Let $\boldsymbol{f} \in \boldsymbol{V}'$ be arbitrary. Any solution $\boldsymbol{v} \in \boldsymbol{V}$ of* (2.9) *satisfies*

$$\|\boldsymbol{v}\| \leq \nu^{-1/2} \|\boldsymbol{f}\|_{-1} \,. \tag{3.1}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{V}$ be a solution of (2.9). We take in (2.9) as test function $\boldsymbol{\varphi} = \boldsymbol{v}$, and obtain

$$\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\boldsymbol{v}\big) + \nu(\nabla\boldsymbol{v},\nabla\boldsymbol{v}) - \frac{1}{2}\int_{S_1}(\boldsymbol{v}\cdot\boldsymbol{n})_-|\boldsymbol{v}|^2 d\sigma = \langle \boldsymbol{f},\boldsymbol{v}\rangle.$$

Due to (2.6) we get

$$\begin{aligned}\|\boldsymbol{v}\|^2 &\leq \frac{1}{2}\int_{S_1}(\boldsymbol{v}\cdot\boldsymbol{n})_+|\boldsymbol{v}|^2 d\sigma + \nu\|\nabla\boldsymbol{v}\|^2 \\ &= \langle \boldsymbol{f},\boldsymbol{v}\rangle \\ &\leq \nu^{-1/2}\|\boldsymbol{f}\|_{-1}\|\boldsymbol{v}\| \,.\end{aligned}$$

This implies the desired inequality. □

This energy estimate is the background of the construction of weak solutions.

Theorem 3.1. *Let Ω a region in $\mathbb{R}^d$, $d \in \{2,3\}$, and $S_1 \subset \partial\Omega$ uniform C^1-regular. For every $\boldsymbol{f} \in H^{-1}(\Omega)^d$ there exists a weak solution to problem* (2.9) *satisfying* (3.1).

Proof. We apply the Galerkin method, see [7, 13], and consider approximative solutions by solving reduced problems projected on finite dimensional subspaces $\boldsymbol{V}_n \subset \boldsymbol{V}$. Find $\boldsymbol{v}_n \in \boldsymbol{V}_n$ such that

$$\big((\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n,\boldsymbol{\varphi}\big) + \nu(\nabla\boldsymbol{v}_n,\nabla\boldsymbol{\varphi}) - \frac{1}{2}\int_{S_1}(\boldsymbol{v}_n\cdot\boldsymbol{n})_-\boldsymbol{v}_n\boldsymbol{\varphi}\, d\sigma \quad = \quad \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi}\in\boldsymbol{V}_n. \tag{3.2}$$

The properties of the energy method guarantee the bound (3.1) for the sequence,

$$\|\boldsymbol{v}_n\| \quad \leq \quad \nu^{-1/2}\|\boldsymbol{f}\|_{-1} \,.$$

Since the right hand side of this bound is independent of n, we are able to find a sub-sequence which weakly converges to a function $\boldsymbol{v} \in \boldsymbol{V}$. Hence, denoting this subsequence also by $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$, we get the weak convergence $\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}$ in $H^1(\Omega)^d$. It remains to show, that $\boldsymbol{v}$ is a (weak) solution of (2.9). This is obtained by showing the following regularity features. Due to the compact embedding $H^1(\Omega) \subset\subset L^2(\Omega)$, we have (cf. [19])

$$\boldsymbol{v}_n \to \boldsymbol{v} \qquad \text{strongly in } L^2(\Omega)^d \,.$$

Further, due to the continuous embedding $H^1(\Omega) \subset L^q(\Omega)$ for $1 \leq q \leq 6$, we have

$$(\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n \rightharpoonup (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \qquad \text{weakly in } L^2(\Omega)^d.$$

This ensures the convergence of the volume integrals. A key point of our system is the boundary term which can be handled by a trace theorem, see Adams [1], showing the continuos embedding $\gamma : H^1(\Omega) \to L^q(S_1)$ with

$$\begin{aligned} d = 2: &\qquad 2 \leq q < \infty, \\ d = 3: &\qquad 2 \leq q \leq 4\,. \end{aligned}$$

This ensures $\boldsymbol{v}_n|_{S_1} \in L^q(S_1)^d$ for the corresponding q, and hence also, $(\boldsymbol{v}_n \cdot n)|_{S_1} \in L^q(S_1)$. The embedding is even compact for $2 \le q < 4$. Hence, we find

$$\boldsymbol{v}_n|_{S_1} \to \boldsymbol{v}|_{S_1} \quad \text{and} \quad (\boldsymbol{v}_n \cdot \boldsymbol{n})_-|_{S_1} \to (\boldsymbol{v} \cdot \boldsymbol{n})_-|_{S_1} \quad \text{in} \quad L^q(S_1)^d \quad \text{for} \quad 2 \le q < 4. \tag{3.3}$$

This implies for the product

$$(\boldsymbol{v}_n \cdot \boldsymbol{n})_- \boldsymbol{v}_n|_{S_1} \quad \to \quad (\boldsymbol{v} \cdot \boldsymbol{n})_- \boldsymbol{v}|_{S_1} \quad \text{strongly in in } L^q(S_1)^d \quad \text{for} \quad 1 \le q < 2\,.$$

The $H^1(\Omega)$ bound of $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ yields that the sequence $(\boldsymbol{v}_n|_{\partial\Omega})_{n\in\mathbb{N}}$ is estimated in $H^{1/2}(\partial\Omega)$. In the most restrictive case, for $d = 3$, the imbedding theorem ensures that $H^{1/2}(\partial\Omega) \subset L^4(\partial\Omega)$. Moreover $H^{1/2}(\partial\Omega) \subset L^q(\partial\Omega)$ compactly for $1 < q < 4$. Therefore, the limit of the product is a product of weak limits:

$$(\boldsymbol{v}_n \cdot \boldsymbol{n})_- \boldsymbol{v}_n|_{S_1} \quad \rightharpoonup \quad (\boldsymbol{v} \cdot \boldsymbol{n})_- \boldsymbol{v}|_{S_1} \quad \text{weakly in} \quad L^2(S_1)^d\,.$$

Taking $\boldsymbol{\varphi} \in \boldsymbol{V}$, we pass to the limit $n \to \infty$ and obtain that $\boldsymbol{v}$ is a weak solution of (2.9). □

It is worthwhile to underline that we are not able to prove uniqueness of stationary solutions. Similar to the pure Dirichlet case, see [7], this can be done just for small forces (in particular for $\boldsymbol{f} = 0$), which is not possible for the classical do-nothing condition.

In the following we show that the boundary condition (2.10) can be recovered from the weak formulation.

Theorem 3.2. *Assume that $f \in L^2(\Omega)^d$ and a solution given by Theorem 3.1 is smooth, at least $\boldsymbol{v} \in H^2(\Omega)^d$. Then $\boldsymbol{v}$ fulfills the DDN boundary condition* (2.10).

Proof. Let $\boldsymbol{v} \in H^2(\Omega)^d$ be a solution of (2.9). Then it holds for all $\boldsymbol{\varphi} \in \boldsymbol{V}$

$$\big((\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} - \boldsymbol{f}, \boldsymbol{\varphi}\big) + \int_{S_1} \left(-\frac{1}{2}(\boldsymbol{n} \cdot \boldsymbol{v})_- \boldsymbol{v} + \nu(\nabla\boldsymbol{v} \cdot \boldsymbol{n}) \right) \boldsymbol{\varphi}\, d\sigma = 0. \tag{3.4}$$

In particular, (3.4) is fulfilled for $\boldsymbol{\varphi} \in C_0^\infty(\Omega)$ with $\operatorname{div} \boldsymbol{\varphi} = 0$. In that case, $\boldsymbol{\varphi}|_{S_1} = 0$, so that the boundary integral vanishes.

$$\big((\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} - \boldsymbol{f}, \boldsymbol{\varphi}\big) = 0 \qquad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega)^d \cap \boldsymbol{V}\,.$$

The theory of Helmholtz decomposition [19] implies the existence of a scalar function $p \in C^1(\Omega)$ such that

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} - \boldsymbol{f} + \nabla p = 0 \quad \text{ in } \Omega. \tag{3.5}$$

The regularity of p is controlled by the theory, too. Hence, the above identity holds for all points. Furthermore, for arbitrary $\boldsymbol{\varphi} \in \boldsymbol{V}$ there holds

$$(\nabla p, \boldsymbol{\varphi}) - \int_{\partial\Omega} p(\boldsymbol{n} \cdot \boldsymbol{\varphi})\, d\sigma = 0. \tag{3.6}$$

Adding (3.6) to (3.4) we get for all $\boldsymbol{\varphi} \in V$

$$\big((\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla p - \boldsymbol{f}, \boldsymbol{\varphi}\big) + \int_{S_1} \left(-\frac{1}{2}(\boldsymbol{n} \cdot \boldsymbol{v})_- \boldsymbol{v} + (\nu\nabla\boldsymbol{v} - pI) \cdot n \right) \boldsymbol{\varphi} d\sigma = 0. \tag{3.7}$$

But by (3.5) the first term vanishes. Hence the boundary term is zero for all $\boldsymbol{\varphi} \in \boldsymbol{V}$. We conclude the validity of the boundary condition (2.10). □

Remark 3.1. The additional boundary term in (2.9) is nonlinear. Hence, a natural question is whether this system is easier or more difficult to solve than (2.5). The additional boundary term is only non-zero in the case of inflow on S_1 ($\boldsymbol{v} \cdot n < 0$). Moreover, the term is differentiable and quadratic in $\boldsymbol{v}$. Therefore, usual Newton-type solvers are easily applicable to solve the nonlinear system. In the numerical tests below it even turns out, that the system may be easier to solve than (2.5).

4. Evolutionary System

The Navier-Stokes system with the CDN condition is much better studied in the evolutionary case than in the steady case, [10]. The reason is the lack of energy estimate as in Lemma 3.1 for the CDN condition. Therefore, it is natural to also consider the time-dependent case in combination with the DDN condition:

$$\begin{aligned} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \operatorname{div} \mathbb{T}(\boldsymbol{v}, p) = \boldsymbol{f} &\quad \text{in } \Omega \times (0,T), \\ \operatorname{div} \boldsymbol{v} = 0 &\quad \text{in } \Omega \times (0,T), \\ \boldsymbol{v} = 0 &\quad \text{at } S_0 \times (0,T), \\ \mathbb{T}(\boldsymbol{v}, p) \cdot \boldsymbol{n} - \frac{1}{2}(\boldsymbol{v} \cdot \boldsymbol{n})_- \boldsymbol{v} = 0 &\quad \text{at } S_1 \times (0,T), \\ \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 &\quad \text{on } \Omega. \end{aligned}$$

In the weak formulation we seek $\boldsymbol{v} \in L^\infty(0,T; L^2(\Omega)^d) \cap L^2(0,T; \boldsymbol{V})$ such that

$$(\partial_t \boldsymbol{v}, \boldsymbol{\varphi}) + \big((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{\varphi}\big) + \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - \frac{1}{2}\int_{S_1} (\boldsymbol{v} \cdot \boldsymbol{n})_- \boldsymbol{v}\boldsymbol{\varphi}\, d\sigma = (\boldsymbol{f}, \boldsymbol{\varphi}) \tag{4.1}$$

in distributional meaning on the time interval $[0,T)$ for each divergence free $\boldsymbol{\varphi} \in C^\infty(\overline{\Omega} \times [0,T], \mathbb{R}^d)$ such that $\boldsymbol{\varphi} = \boldsymbol{0}$ on $(S_0 \times [0,T]) \cup (\Omega \times \{T\})$.

Theorem 4.1. *Let $\boldsymbol{f} \in L^2(0,T; V^*)$, $\boldsymbol{v}_0 \in L^2(\Omega)^d$, div $\boldsymbol{v}_0 = 0$ in $\mathcal{D}'(\Omega)$. Then there exists at least one distributional solution $\boldsymbol{v} \in L^\infty(0,T; L^2(\Omega)^d) \cap L^2(0,T; \boldsymbol{V})$ of (4.1). Furthermore, $\boldsymbol{v}$ fulfills the stability property*

$$\|\boldsymbol{v}\|_{L^\infty(0,T;L^2(\Omega)^d)} + \nu\|\boldsymbol{v}\|_{L^2(0,T;H^1(\Omega)^d)} + \int_0^T\int_{S_1} (\boldsymbol{v} \cdot \boldsymbol{n})_+ |\boldsymbol{v}|^2 d\sigma dt \le C\,,$$

with a data dependent constant $C = C(T, \boldsymbol{f}, \boldsymbol{v}_0)$.

Proof. The proof is obtained by classical techniques from the theory of the Navier-Stokes equations [19]. We repeat the considerations from the proof of Theorem 3.1, but in its evolutionary version. Let $\{\boldsymbol{w}_k : k \in \mathbb{N}\}$ be an orthonormal basis of $\boldsymbol{V}$, i.e., $(\boldsymbol{w}_k, \boldsymbol{w}_l) = \delta_{kl}$. We are looking for the solution as a limit of an approximative sequence $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$ of the form

$$\boldsymbol{v}_n = \sum_{k=1}^n a_n^{(k)}(t)\boldsymbol{w}_k(x)\,.$$

We have to show that such time depending parameters $\{a_n^{(k)}(t)\}$ for $k = 1, ..., n$ exist. For fixed n we consider the following ODE systems

$$(\partial_t \boldsymbol{v}_n, \boldsymbol{w}_k) + (v_n \cdot \nabla \boldsymbol{v}_n, \boldsymbol{w}_k) + \nu(\nabla \boldsymbol{v}_n, \nabla \boldsymbol{w}_k) - \frac{1}{2}\int_{S_1} (\boldsymbol{v}_n \cdot n)_- \boldsymbol{v}_n \boldsymbol{w}_k d\,\sigma = (\boldsymbol{f}, \boldsymbol{w}_k) \tag{4.2}$$

for all $k \in \{1, \ldots, n\}$. The Picard theorem implies the existence of the system, but in general the time of existence T_n may highly depend on the number n. In order to overcome this obstacle, we need an a priori estimate. Diagonal testing of (4.2) leads to the following bound

$$\sup_{t\in[0,T]} \|\boldsymbol{v}_n(t)\|^2 + \nu \int_0^T \|\nabla \boldsymbol{v}_n(t)\|^2 dt + \int_0^T \int_{S_1} (\boldsymbol{v}_n \cdot \boldsymbol{n})_+ |\boldsymbol{v}_n|^2 d\sigma dt$$
$$\leq C \left(\int_0^T \|\boldsymbol{f}(t)\|^2 dt + \|\boldsymbol{v}_0\|^2_{L^2(\Omega)} \right).$$

This bound gives us the possibility to consider the approximative solutions on the same time interval, which in our case is an arbitrary $T > 0$. From the bound we get that

$$\begin{array}{lll} \boldsymbol{v}_n & \text{is uniformly bounded in} & L^2(0,T;H^1(\Omega)^d), \\ \boldsymbol{v}_n & \text{is uniformly bounded in} & L^\infty(0,T;L^2(\Omega)^d), \\ \partial_t \boldsymbol{v}_n & \text{is uniformly bounded in} & L^2(0,T;H^{-1}(\Omega)^2), \\ & \text{and in} & L^2(0,T;H^{-3/2}(\Omega)^3). \end{array} \tag{4.3}$$

The bounds (4.3) allows us to find a subsequence (also denoted by $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$) such that

$$\begin{array}{rcll} \boldsymbol{v}_n & \rightharpoonup & \boldsymbol{v} & \text{weakly in } L^2(0,T;H^1(\Omega)^d)\,, \\ (\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n & \rightharpoonup & (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} & \text{weakly in } L^{5/4}(\Omega \times (0,T))\,. \end{array}$$

By classical considerations one can show that $\boldsymbol{v}$ fulfills (4.1). We skip the details since it is standard. □

5. Numerical Results

In this section, we illustrate the effect of the do-nothing conditions CDN and DDN for steady flows and non-steady flows. Let $\mathcal{T}_h$ be a shape-regular, admissible decomposition of Ω into quadrilaterals. Let $\hat{K} := (-1,1)^d$ denote the reference element and $\mathbb{Q}_r(\hat{K})$ the space of all polynomials on $\hat{K}$ with maximal degree $r \geq 0$ in each coordinate direction. By $F_K : \hat{K} \to K$ we denote a bilinear mapping from the reference cell to K. We use the H^1-conforming biquadratic finite elements ($r = 2$),

$$Q_r(\mathcal{T}_h) := \left\{ v \in H^1(\Omega) : \; v|_K \circ F_K \in \mathbb{Q}_r(\hat{K}) \quad \forall K \in \mathcal{T}_h \right\}.$$

The discrete velocity space V_h and discrete pressure space Q_h are given by:

$$\boldsymbol{V}_h = Q_2(\mathcal{T}_h)^d \cap \left\{ \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v}|_{S_0} = 0 \right\},$$
$$Q_h = Q_2(\mathcal{T}_h)\,.$$

Since the equal-order finite element pair $\boldsymbol{V}_h \times Q_h$ is known to be not inf-sup stable, we add local projection stabilization terms to the discrete formulation. For stabilizing the convective term we also use additional local projection terms. For details we refer to [2]. In the stationary case, the discrete system with DDN condition seeks $\boldsymbol{v}_h \in \boldsymbol{V}_h,\ p_h \in Q_h$ such that:

$$\big((\boldsymbol{v}_h \cdot \nabla)\boldsymbol{v}_h, \boldsymbol{\varphi}\big) + (\nu \nabla \boldsymbol{v}_h, \nabla \boldsymbol{\varphi}) - (p, \operatorname{div} \boldsymbol{\varphi}) + (\operatorname{div} \boldsymbol{v}_h, \xi)$$
$$- \frac{1}{2} \int_{S_1} (\boldsymbol{v} \cdot \boldsymbol{n})_- \boldsymbol{v} \boldsymbol{\varphi} \, d\sigma + s_h(\boldsymbol{v}_h, p_h; \boldsymbol{\varphi}, \xi) = (\boldsymbol{f}, \phi)$$

for all $(\boldsymbol{\varphi}, \xi) \in \boldsymbol{V}_h \times Q_h$. For the definition of the stabilization term s_h we refer to [3]. The formulation with the CDN condition is similar but without the boundary term.

5.1. Steady flow

As first test problem we take the unit square $\Omega := (0,1)^2$ with do-nothing boundary at $S_1 := \{0\} \times (0,1)$ and homogeneous Dirichlet data on $S_0 := \partial\Omega \setminus S_1$. The right hand side is chosen as $\boldsymbol{f}(x,y) = (\sin(x) + \sin(y), 0)$. It generates a vortex with clockwise orientation. We compare the flow obtained with the CDN condition (2.4) and with the proposed DDN condition (2.9). In this example, it is a priori not clear which part of S_1 is an inflow part and which part is an outflow part. Hence, we expect a difference for the two types of boundary conditions (2.4) and (2.9).

The obtained flow fields are shown in Fig. 5.1 for four values of viscosity $\nu = 5 \cdot 10^{-k}$, $k \in \{1,2,3,4\}$. The lines represent the streamlines of the velocities and the color the pressure field. The upper part of the left boundary becomes an inflow boundary and the lower part of the left boundary becomes an outflow boundary. For smaller viscosities the inflow part increases and the flow is more accelerated. Hence, both do-nothing boundary conditions allows for in- and outflow simultaneously. The flow field obtained with the CDN condition (2.4) and with the proposed DDN condition (2.9) are very similar for low Reynolds numbers. For larger Reynolds numbers, the differences become more obvious. Moreover, the linear (and nonlinear) solver did not converge with the classical do-nothing (CDN) condition and viscosity $\nu \leq 5 \cdot 10^{-4}$. Probable explanations are the absence of a stationary solution due to the amount of inflow or the reduced convergence radius of the Newton solver. For the directional do-nothing (DDN) condition, solving the nonlinear equations was much easier. Table 5.1 shows the number of

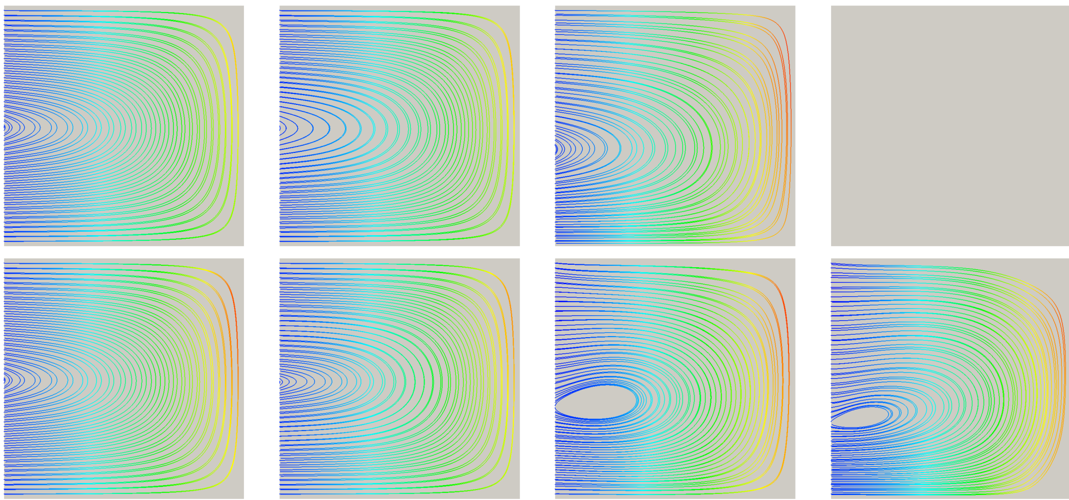

Fig. 5.1. Stationary streamlines obtained with classical do-nothing condition (CDN) (upper row) and directional do-nothing condition (DDN) (lower row) at the left boundary and for different viscosities $\nu = 0.5$ (left), $\nu = 0.05$ (middle) and $\nu = 0.005$ (right). The colors show the pressure field. The two different boundary conditions lead to very similar flow pattern for low Reynods number. For higher Reynolds number, the differences become larger. In particular, the DDN condition has a small dissipative effect onto the inflow. For $\nu = 10^{-4}$ and CDN condition, the solver failed to compute a stationary solution.

Newton iterations and linear solves for the two methods CDN and DDN in dependence of the viscosity. These numbers correspond to the solver performance on the mesh with 16,384 cells and a reduction of the residuum by a factor of 10^{-6}. The number of linear solvers are related to the total number of multigrid iterations in the Newton algorithm. Since the linear problems in the Newton algorithm are not needed to be solved exactly, the linear tolerance is 10^{-2}. The table shows that the number of linear iterations are in the range of 3-9 iterations per Newton step. For smaller viscosities the number of Newton steps increase moderately. We do not observe any drawbacks for DDN. In the contrary, when the viscosity becomes smaller less linear solves are necessary with DDN than with CDN. For viscosities smaller or equal to 0.0005, the linear solver diverges for the classical do-nothing condition CDN. This shows that the solver may benefit from the DDN condition in the case that the (artificial) boundary S_1 is not a pure outflow boundary.

Table 5.1: Linear inflow flux j_1, nonlinear outflow flux j_2, and comparison of nonlinear and linear solves for the do-nothing and the directional do-nothing condition in dependence of the viscosity for the example in sect. 5.1.

ν	0.5	0.05	0.005	0.0005	0.0002
CDN					
inflow flux j_1	-4.510e-3	-4.498e-2	-1.887e-1	|	|
nonlinear outflow flux j_2	6.10e-7	6.109e-4	7.354e-2	*divergence*	*divergence*
nonlinear/linear iterations	3/10	3/19	3/27	|	|
DDN					
inflow flux j_1	-4.507e-3	-4.269e-2	-1.593e-1	-1.900e-1	-1.942e-1
nonlinear outflow flux j_2	6.10e-7	5.318e-4	4.712e-2	1.207e-1	1.372e-1
nonlinear/linear iterations	3/10	3/19	3/25	6/16	9/27

To characterize the difference of the solutions more quantitatively we consider two type of functional output, the inflow flux j_1 and the nonlinear outflow flux j_2, defined by

$$j_1(\boldsymbol{v}) := \int_{S_1} (\boldsymbol{v} \cdot \boldsymbol{n})_- \, d\sigma \,, \quad \text{and} \quad j_2(\boldsymbol{v}) \ := \ \int_{S_1} (\boldsymbol{v} \cdot \boldsymbol{n})_+ \boldsymbol{v}^2 \, d\sigma \,.$$

Note that the total flux is zero for both do-nothing conditions, because $\int_{S_1} (\boldsymbol{v}\cdot\boldsymbol{n}) \, d\sigma = \int_\Omega \operatorname{div} \boldsymbol{v} = 0$. The obtained values are also given in Table 5.1. The two boundary conditions lead to the same fluxes for low Reynolds number. But for increasing Reynolds number the fluxes deviate more. In particular, the absolute value of the linear inflow flux j_1 and of the nonlinear outflow flux j_2 are both reduced for the directional do-nothing condition (2.9). The reduction depends on the Reynold number and becomes more obvious for increasing Reynolds number. This behavior can be explained by the stability property (3.1) which holds for (2.9) but not for (2.4). Hence, the introduced DDN condition (2.9) leads to more dissipation in the inflow flux. This is indeed a desirable detail, because such kind of boundary conditions are usually used for pure outflow. This can also be important from the numerical point of view, because in this example at higher Reynolds number ($\nu \leq 5 \cdot 10^{-4}$), the stationary solver did not converge for the CDN condition.

5.2. Backward facing step

A standard problem in computational fluid dynamics consists of a flow behind a backward facing step. The computational domain is $\Omega = [(-1, 0] \times (0, 1)] \cup [(0, L) \times (-1, 1)]$ and the

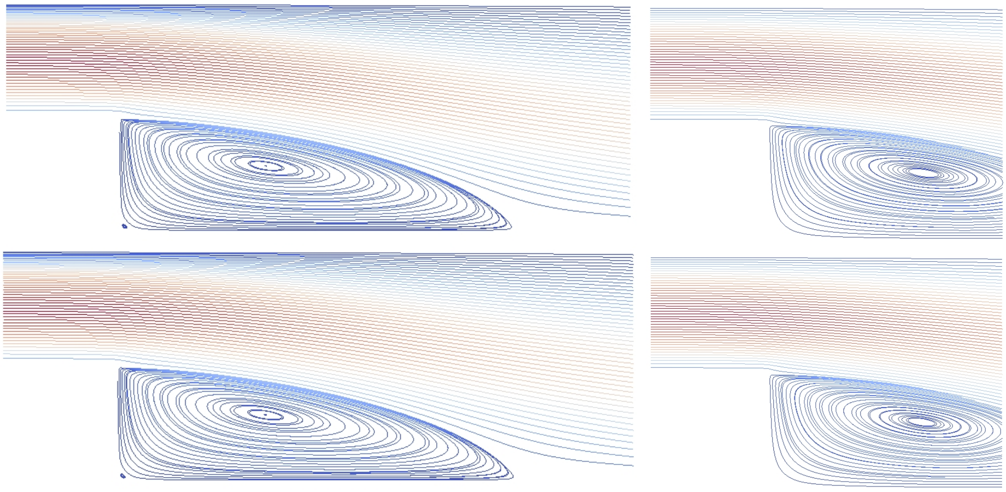

Fig. 5.2. Streamlines of the backfard facing stepflow at Reynolds number $Re = 100$ for two domains with different lenght, $L = 4.5$ (left) and $L = 2$ (right): The differences of the CDN (upper row) and the DDN condition (lower row) are marginal.

ouflow boundary is $S_1 = \{L\} \times (-1, 1)$. A parabolic inflow is located at $x = -1$. We compare the CDN and the DDN condition for different positions $0 < L < 6$ for the ouflow boundary. At Reynolds number $Re = 100$, the center of the main vortex is located at $(x, y) \approx (1.5, -0.4)$. In Fig. 5.2 the streamlines of the flow for the two positions $L = 2$ and $L = 4$ are shown. For the short configuration $L = 2$, the outflow boundary condition is located inside the main vortex. However, the center of the vortex is only marginally influenced for both types of boundary conditions.

In order to make the comparison more precise, we take a look on the horizontal drag force

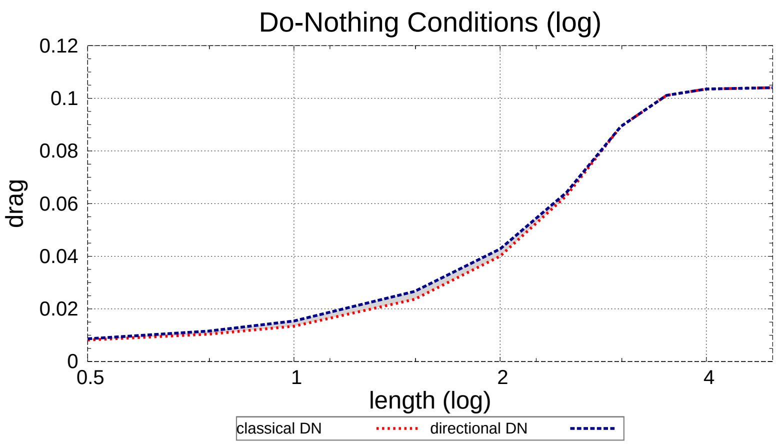

Fig. 5.3. Drag c_{drag} in dependence of the length of domain L in the backward facing step configuration. The blue curve shows the drag for the DDN condition, the red curve corresponds to the CDN condition. For $L \geq 4$ the drag becomes independent of L. Both conditions behave very similar. The DDN perfoms yet slightly better.

of the flow on the vertical boundary part, $\Gamma := 0 \times (-1, 0)$, at the step:

$$c_{drag} = \int_\Gamma \left(\nu \frac{\partial v_x}{\partial x} - p n_x \right) ds\,.$$

The dependence of this quantity is expected to converge if the length L of the domain goes to infinity, $L \to \infty$. In Fig. 5.3 the drag is plotted for the two types of boundary conditions. For $L \geq 4$ the drag value is rougly speaking not affected by the precise value of L. For smaller values of L, the vortex becomes intersected with the outflow boundary S_1. Values of $L < 1.5$ imply that the center of the vortex is even located outside the domain. Hence, the drag cannot be computed accurately enough. The drag value reduces for smaller values of L. However, both conditions CDN and DDN behave very similar. As shown in Fig. 5.3, the DDN perfoms yet marginally better.

5.3. Van Karman vortex street

For the non-steady case, we take as numerical example the geometry "Flow around a circular cylinder" (see [17]). The left boundary is an inflow boundary with parabolic profile. Lower and upper walls, as well as the obstacle are no slip conditions (homogeneous Dirichlet). On the right boundary the natural outflow conditions are imposed. The Reynolds number is taken higher

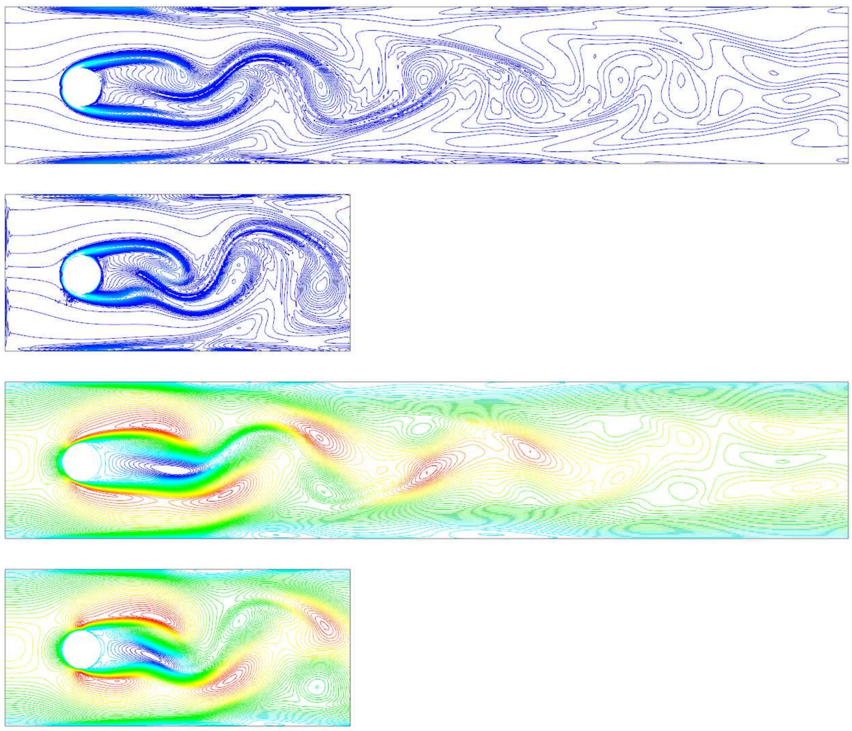

Fig. 5.4. Directional do-nothing outflow condition: Vorticity rot $\boldsymbol{v}$ (left) and iso-lines of the horizontal velocity component (right), v_1, after 150 time steps in two domains with different tube length. The influence of the directional do-nothing outflow condition seems to be restricted to the neighborhood of the right boundary and has little global impact. No differences can be observed compared to the classical do-nothing condition (not shown here).

as in the original benchmark configurations, namely $Re = 300$, in order to be in the dynamic regime. The flow is non-stationary with the typical Von-Karman vortices forming behind an obstacle.

We performed four simulations with two different tube length behind the obstacle and both types of natural outflow condition (CDN and DDN). The information is mainly convected downstream. For an ideal outflow boundary condition, the solution should be independent of the tube length. However, a small impact of the boundary condition will not be avoidable due to a certain information transport also in upstream direction.

In Fig. 5.4 the vorticity $\operatorname{rot} \boldsymbol{v} = \partial_y v_1 - \partial_x v_2$ and the mean velocity component v_1 are shown for the simulations with the DDN condition after 150 time steps. Obviously, the impact of the natural outflow condition is not negligible but relatively small. Some differences can be observed behind the obstacle. However, the results with the CDN condition become absolutely the same (up to machine precision). Therefore, we pass to show the corresponding figures. The reason for the equality is the fact that both boundary conditions coincide for pure outflow $(\boldsymbol{v} \cdot \boldsymbol{n}|_{S_1} \geq 0)$, which is the case for this van Karman vortex street.

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