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MODELING THE LID DRIVEN FLOW: THEORY AND COMPUTATION

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Abstract. Motivated by the study of the corner singularities in the so-called cavity flow, we establish in the first part of this article, the existence and uniqueness of solutions in $L^2(\Omega)^2$ for the Stokes problem in a domain Ω, when Ω is a smooth domain or a convex polygon. This result is based on a new trace theorem and we show that the trace of u can be arbitrary in $L^2(\partial\Omega)^2$ except for a standard compatibility condition recalled below. The results are also extended to the linear evolution Stokes problem. Then in the second part, using a finite element discretization, we present some numerical simulations of the Stokes equations in a square modeling thus the well known lid-driven flow. The numerical solution of the lid driven cavity flow is facilitated by a regularization of the boundary data, as in other related equations with corner singularities ([9], [10], [45], [24]). The regularization of the boundary data is justified by the trace theorem in the first part.

Key words. Stokes and related (Oseen, etc.) flows, weak solutions, existence, uniqueness, regularity theory, lid driven cavity.

Introduction

We are interested in the first part of this article in the existence of L^2-solutions for the (linear stationary) Stokes problem in a domain Ω of $\mathbb{R}^2$. The set Ω is assumed to be bounded, regular of class $\mathcal{C}^2$, or it could be a convex polygonal domain. More generally Ω can be what we will call a (convex) *domain of polygonal type* that is Ω a piecewise $\mathcal{C}^2$ domain for which the tangent to the boundary Γ has a finite number of discontinuity points $S_1, \ldots, S_N$, with a well defined left and right tangent at these points, the angle between the tangents being $0 < \alpha_j < \pi$; hence the domain Ω needs not be convex in this case, we only require the angles to be convex.

Motivated by the study of a flow in such a domain Ω (see Part 2) we start with the stationary linear Stokes problem, which, in its most general form reads:

$$(1)\qquad \begin{cases} -\Delta\tilde{u} + \nabla\tilde{p} = f \quad \text{in} \quad \Omega, \\ \text{div}\ \tilde{u} = h \quad \text{in} \quad \Omega, \\ \tilde{u} = g \quad \text{on} \quad \Gamma = \partial\Omega. \end{cases}$$

The emphasis in the second part of this article is on the so-called lid-driven cavity problem. In this case Ω is the square $(0,1)\times(0,1), f = h = 0$ and $g = (0,0)$ at $x = 0, 1$, and $y = 0$, and $g = (1,0)$, at $y = 1$; the discontinuities of g produce singularities and vortices at the corners $(0,1)$ and $(1,1)$. We describe this example in more detail in Section 3 and in Part 2 where we exhibit some numerical simulations related to the lid-driven cavity problem using the classical finite element discretization method together with a regularization of the boundary values of the velocity justified by the results in Part 1.

We know that if $f \in L^2(\Omega)^2, h = g = 0$, then the existence and uniqueness of a solution $\tilde{u} \in H_0^1(\Omega)^2$ of (1) is derived from the projection theorem, and $\tilde{p} \in L^2(\Omega)$

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follows from the result of Deny-Lions [13]; see also [59], [60]. If $f \in L^2(\Omega)^2$ and $h \in L^2(\Omega)$, with $\int_\Omega h dxdy = 0$, we have existence and uniqueness of $U \in H^2(\Omega)^2, P \in H^1(\Omega)$ satisfying

$$\begin{cases} -\Delta U + \nabla P = f \quad \text{in} \quad \Omega, \\ \text{div } U = h \quad \text{in} \quad \Omega, \\ U = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{2}$$

In the case where Ω is smooth, this result is proved in [8]; see also [31]. When Ω is of polygonal type, this result is proven in [32]; see also [33]. The difference $u = \tilde{u} - U, p = \tilde{p} - P$ is solution of the following problem which concentrates the lack of regularity on the boundary value, like for the lid driven cavity problem:

$$\begin{cases} -\Delta u + \nabla p = 0 \text{ in } \Omega, \\ \text{div } u = 0 \text{ in } \Omega, \\ u = g \text{ on } \Gamma. \end{cases} \tag{3}$$

In Section 1 we derive a trace theorem for functions u in $L^2(\Omega)^2$ satisfying $(3)_1$, $(3)_2$, thus giving a meaning to $(3)_3$. Then in Section 2 we establish the existence and uniqueness of a solution $u \in L^2(\Omega)^2$ of (3) provided $\int_\Gamma g \cdot n \, d\Gamma = 0$, see Theorem 3; here n is the unit outward normal vector on Γ and below τ is the unit tangent vector, such that n is directly orthogonal to τ. We discuss in more details an example of lid driven cavity flow in Section 3. Finally in Section 4 we extend the results to the linear evolution Stokes problem. Namely, we prove the necessary trace theorems, then, in Theorem 5, we show that if g is given in $L^2(0,T;L^2(\Gamma)^2)$ satisfying $\int_\Gamma g \cdot n \, d\Gamma = 0$ for a.e. $t \in (0,T)$, then there exists a unique solution $u \in L^2(0,T;L^2(\Omega)^2)$ of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = 0 \text{ in } (0,T) \times \Omega, \\ \text{div } u = 0 \text{ in } \Omega, \\ u = g \text{ on } (0,T) \times \Gamma, \\ u(0) = 0, \end{cases} \tag{4}$$

with $p \in \mathcal{D}'((0,T) \times \Omega)$.

There is a large literature on weak solutions of the Navier-Stokes equations starting with the now classical results of J. Serrin [53] of solutions of the Navier-Stokes equations in $L^r(0,T;L^s)$ up to the recent result [27] of uniqueness of solutions in $L^3(0,T;L^3(\mathbb{R}^3)^3)$, itself followed by a series of articles improving and simplifying its proof.

In another direction Fabes, Kenig and Verchota study in [18] the stationary Stokes problem on Lipschitz domains of $\mathbb{R}^n, n \geq 3$, using methods of harmonic analysis. This work is extended to the time-dependent case in the articles [52], [51] which study the evolution Stokes equation in a Lipschitz domain of $\mathbb{R}^n, n \geq 3$, in L^2 space in [52] and in some L^p spaces in [51].

In yet another direction, H. Amann introduced in [2] and [1] a concept of weak solutions for the Navier-Stokes equations using the concept of duality or adjoint equations. The idea is to integrate by parts all or most of the derivatives against a smooth test function. The whole book [41] is devoted to constructing such weak solutions of linear elliptic and parabolic equations, but they do not address the case of the Stokes equations. The work of [1,2] has been subsequently extended in different directions (Stokes vs Navier-Stokes, stationary vs time dependent, space L^2 vs spaces L^p) in a series of articles [22], [21], [23], [20] ; see also [46], [43]. These