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AN AUGMENTED IIM & PRECONDITIONING TECHNIQUE FOR JUMP EMBEDDED BOUNDARY CONDITIONS

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Abstract. A second-order accurate augmented method is proposed and analyzed in this paper for a general elliptic PDE with a general boundary condition using the jump embedded boundary conditions (JEBC) formulation. First of all, the existence and uniqueness of an interface problem with given are discussed. Then, the well-posedness theory is extended to the interface problems with given jump conditions. In the proposed numerical method, one novel idea is to preconditioning the PDE first so that the coefficient of the highest derivative is of $O(1)$. The second idea is to introduce two augmented variables corresponding to the jump in the solution and its normal derivative along the boundary to get an interface problem. For a piecewise constant coefficient, the fast Poisson solver then can be utilized in a rectangular domain. The augmented variables can be determined from a Schur complement system. We also propose two preconditioning techniques for the GMRES iterative method for the Schur complement; one is from the flux jump condition, and the other one is from the algebraic preconditioner based on the interpolation scheme in the augmented algorithm. The presented numerical results show that the proposed method has not only obtained second order accurate solutions in the L^∞ norm globally, but also second order accurate normal derivatives at the boundary from each side of the interface. The proposed preconditioning technique can speed up 50-90% compared with the method without preconditioning.

Key words. Jump embedded boundary conditions (JEBC), augmented immersed interface method, fast Poisson solver, irregular domain, PDE and algebraic preconditioner.

1. Introduction and mathematical formulation

Let $\Omega \subset \mathbb{R}^d$ ($d=2$ or 3 in practice) be an open bounded and connected set with a Lipschitz continuous boundary $\Gamma := \partial\Omega$, and $\boldsymbol{\nu}$ be the outward unit normal vector on Γ. The domain Ω is composed of two disjoint Lipschitz subdomains, the interior domain Ω^- and the exterior one Ω^+ separated by a Lipschitz continuous interface $\Sigma \subset \mathbb{R}^{d-1}$ such that $\Omega = \Omega^- \cup \Sigma \cup \Omega^+$, as shown in Figure 1. The extension to several closed interfaces is straightforward. The case of more general situations when Σ cuts Γ can be treated as well but it is more technical and we refer to [9] for the trace theory. Let $\boldsymbol{n}$ be the unit normal vector on the interface Σ arbitrarily oriented from Ω^- to Ω^+.

We use the standard notations for the Lebesgue and Sobolev spaces, e.g. [8, 15]. In particular, $\|.\|_{0,\Omega}$ denotes the $L^2(\Omega)$-norm, $\|.\|_{1,\Omega}$ the $H^1(\Omega)$-norm, $\|.\|_{-1,\Omega}$ for the $H^{-1}(\Omega)$-norm, $(.,.)_{0,\Omega}$ for the $L^2(\Omega)$-inner product, and $\langle .,. \rangle_{-1,\Omega}$ for the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$ or $\langle .,. \rangle_{-1/2,\Sigma}$ for the duality pairing between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$. We also define the Hilbert spaces below with their usual

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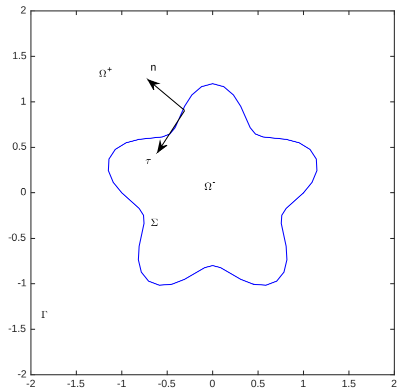

FIGURE 1. Configuration of the interface problem with a closed surface Σ bordering the subdomain Ω^- and $\Omega = \Omega^- \cup \Sigma \cup \Omega^+$. The domain and the interface are used in the numerical experiments.

respective inner products and associated norms:

$$\boldsymbol{H}_{div}(\Omega) = \left\{ \boldsymbol{u} \in L^2(\Omega)^d;\ \boldsymbol{\nabla}\cdot \boldsymbol{u} \in L^2(\Omega) \right\},$$
$$H^1_{0\Gamma}(\Omega) = \left\{ u \in H^1(\Omega);\ u = 0 \text{ on } \Gamma \right\}.$$

For any quantity ψ defined all over Ω, the restrictions on Ω^- or Ω^+ are respectively denoted by $\psi^- := \psi_{|\Omega^-}$ and $\psi^+ := \psi_{|\Omega^+}$. For a function ψ having a jump on Σ, let $\psi^-_{|\Sigma}$ and $\psi^+_{|\Sigma}$ be the traces of ψ^- and ψ^+ on each side of Σ (at least defined in a weak sense), respectively. Following the general framework defined in [3] for scalar elliptic problems with jump interface conditions or its extension to vector problems in [4], let us define the jump of ψ on Σ oriented by $\boldsymbol{n}$ and the arithmetic mean of traces of ψ by :

$$[\![\psi]\!]_\Sigma := \left(\psi^+ - \psi^-\right)_{|\Sigma}, \quad \text{and} \quad \overline{\psi}_\Sigma := \frac{1}{2}\left(\psi^+ + \psi^-\right)_{|\Sigma}.$$

Thus we have also :

$$\psi^+_{|\Sigma} = \overline{\psi}_\Sigma + \frac{1}{2}[\![\psi]\!]_\Sigma, \quad \text{and} \quad \psi^-_{|\Sigma} = \overline{\psi}_\Sigma - \frac{1}{2}[\![\psi]\!]_\Sigma.$$

The interest for choosing such reduced quantities $[\![\psi]\!]_\Sigma$ and $\overline{\psi}_\Sigma$ will appear later with the weak formulation. Let us already notice that when $[\![\psi]\!]_\Sigma = 0$, then we have continuous traces across Σ and $\overline{\psi}_\Sigma = \psi^-_{|\Sigma} = \psi^+_{|\Sigma} = \psi_{|\Sigma}$.

1.1. The classical interface problem with given jumps. The standard scalar elliptic problem considered by Immersed Interface Methods (IIM) [13] reads as