INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 14, Number 4-5, Pages 744–761

A HYBRIDIZABLE WEAK GALERKIN METHOD FOR THE HELMHOLTZ EQUATION WITH LARGE WAVE NUMBER: hp ANALYSIS

JIANGXING WANG AND ZHIMIN ZHANG

Abstract. In this paper, an *hp* hybridizable weak Galerkin (*hp*-HWG) method is introduced to solve the Helmholtz equation with large wave number in two and three dimensions. By choosing a specific parameter and using the duality argument, we prove that the proposed method is stable under certain mesh constraint. Error estimate is obtained by using the stability analysis and the duality argument. Several numerical results are provided to confirm our theoretical results.

Key words. Weak Galerkin method, hybridizable method, Helmholtz equation, large wave number, error estimates.

1. Introduction

In this paper, we develop an *hp*-version hybridizable weak Galerkin (*hp*-HWG) method to solve the Helmholtz equation with Robin boundary condition:

$$\Delta u + \kappa^2 u = \tilde{f} \quad \text{in } \Omega, \tag{1a}$$

$$u = u_0 \quad \text{on } \Gamma_1, \tag{1b}$$

$$\frac{\partial u}{\partial \mathbf{n}} + i\kappa u = \tilde{g} \quad \text{on } \Gamma_2, \tag{1c}$$

where $\Omega \in \mathbb{R}^d$, $d = 2, 3$, is a bounded convex Lipschitz domain, Γ_1 and Γ_2 form a partition of the boundary $\partial\Omega$, $\kappa > 0$ is the wave number, $i = \sqrt{-1}$ is the imaginary unit, and $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$. The condition (1c) is the first order approximation of the radiation condition for Helmholtz scattering problem.

The Helmholtz equation has important applications in electrodynamics, especially in optics and acoustics involving time harmonic wave propagation. The Helmholtz system is not positive definite. When the wave number $\kappa \gg 1$, the solution is highly oscillatory. It is very challenging to design an efficient numerical method to solve the Helmholtz equation with high wave number.

In the literature, there have been extensive investigations devoted to numerical approximations for Helmholtz equations with various boundary conditions. In particular, the finite element method (FEM) has been widely used [3, 7, 17, 18, 21, 22, 35]. It has been shown that the H^1-errors of pth order FEM solutions to the Helmholtz equation have accuracy order $O(\kappa^{p+1}h^p)$ [21, 22, 35, 36]. In [7], Wu et al. analyzed the preasymptotic error of high order FEM and continuous interior penalty FEM (CIP-FEM) for Helmholtz equation with large wave number. They proved that, when $\kappa^{2p+1}h^{2p}$ is sufficiently small, the pollution errors are of order $k^{2p+1}h^{2p}$. Discontinuous Galerkin methods have also been used to solve Helmholtz equations [8, 11, 12, 13, 22, 30]. Detailed analyses have been carried out in [1, 2] on the discrete dispersive relation by *hp*-FEM and high-order discontinuous Galerkin methods. In [28, 29], Shen and Wang used the spectral method to solve

Received by the editors on December 31, 2016, accepted on May 5, 2017.
2000 *Mathematics Subject Classification.* 65N15, 65N30, 35J05.

the Helmholtz equation in both interior and exterior domains. Their results indicate that high-order methods are preferable, if not necessary, for highly oscillatory problems. In [4, 5, 14], hybridizable discontinuous Galerkin methods were used to solve the Helmholtz equation.

The weak Galerkin (WG) method was first introduced by Wang and Ye [32] for second-order elliptic equations. It can be derived from the variational form of the continuous problem by replacing derivatives involved by weak derivatives with some stabilizers. WG methods have been applied to solve many problem [20, 23, 24, 25, 26, 31, 32, 33, 34]. The HWG method [27] was introduced by Mu et al., which applies Lagrange multiplier so that the computational complexity can be significantly reduced.

In this paper, we will develop an *hp*-HWG method to solve the Helmholtz equation with high wave number. The main difficulty in the analysis of the numerical method is due to the strong indefiniteness of the Helmholtz equation. As a consequence, the stability of the numerical approximation is hard to establish. In this work, we use the duality argument to show that the proposed *hp*-HWG method is stable under proper mesh condition. This stability result not only guarantees the existence of the HWG method but also plays an important role in the error analysis. In particular, we first construct an auxiliary problem and establish its *hp*-HWG error estimates; then we combined the estimates with the stability result to derive the error estimates of the *hp*-HWG scheme for the original Helmholtz problem.

Notation. In this paper, standard notations for Sobolev spaces (e.g., $L^2(\Omega)$, $H^k(\Omega)$ for $k \in \mathbb{N}$, etc.) and the associated norms and seminorms will be adopted. Plain and bold fonts are used for scalars and vectors, respectively.

The rest of this paper is organized as follows. The *hp*-HWG scheme for the Helmholtz equation is developed in Section 2. Section 3 is devoted to show the stability result of the numerical scheme. In Section 4, we derive the error estimate of the numerical scheme. Numerical results are given in Section 5 to confirm the theoretical results.

2. Weak Divergence and the *hp*-HWG Scheme

2.1. Weak divergence. Let K be a subdomain in Ω. A *weak vector-valued function* on K refers to a vector field $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$, where $\mathbf{v}_0 \in [L^2(K)]^d$ carries the information of $\mathbf{v}$ in K, and $\mathbf{v}_b \in [L^2(\partial K)]^d$ represents partial or full information of $\mathbf{v}$ on ∂K. It is important to point out that $\mathbf{v}_b$ may not necessarily be related to the trace of $\mathbf{v}_0$ on ∂K, but shall be well-defined. Denote by $\mathbf{V}(K)$ the space of all weak vector-valued functions on K; that is

$$\mathbf{V}(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [L^2(K)]^d, \mathbf{v}_b \in [L^2(\partial K)]^d\}.$$

A weak divergence can be taken for any vector field in $\mathbf{V}(K)$ by following the definition [27].

Definition 2.1. *For any $\mathbf{v} \in \mathbf{V}(K)$, the weak divergence of $\mathbf{v}$, denoted by $\nabla_w \cdot \mathbf{v}$, is defined as a linear functional on $H^1(K)$, whose action on each $\phi \in H^1(K)$ is given by*

$$(\nabla_w \cdot \mathbf{v}, \phi)_K = -(\mathbf{v}_0, \nabla\phi)_K + < \mathbf{v}_b \cdot \mathbf{n}, \phi >_{\partial K}, \tag{2}$$

where $(\cdot,\cdot)_K$ and $< \cdot,\cdot >_{\partial K}$ stand for the inner products in $L^2(K)$ and $L^2(\partial K)$, respectively.

Next, we introduce a discrete weak divergence operator $(\nabla_{w,k}\cdot)$ in a polynomial subspace of the dual of $H^1(K)$ [27].