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Sparse Recovery via ℓ_q-Minimization for Polynomial Chaos Expansions

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Abstract. In this paper we consider the algorithm for recovering sparse orthogonal polynomials using stochastic collocation via ℓ_q minimization. The main results include: 1) By using the norm inequality between ℓ_q and ℓ_2 and the square root lifting inequality, we present several theoretical estimates regarding the recoverability for both sparse and non-sparse signals via ℓ_q minimization; 2) We then combine this method with the stochastic collocation to identify the coefficients of sparse orthogonal polynomial expansions, stemming from the field of uncertainty quantification. We obtain recoverability results for both sparse polynomial functions and general non-sparse functions. We also present various numerical experiments to show the performance of the ℓ_q algorithm. We first present some benchmark tests to demonstrate the ability of ℓ_q minimization to recover exactly sparse signals, and then consider three classical analytical functions to show the advantage of this method over the standard ℓ_1 and reweighted ℓ_1 minimization. All the numerical results indicate that the ℓ_q method performs better than standard ℓ_1 and reweighted ℓ_1 minimization.

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1. Introduction

In the field of uncertainty quantification (UQ), one of the main task is to quantify the effect of uncertain model parameters on model output [14, 24, 26]. When the simulation model is computationally expensive to run, we want to build a cheap surrogate of the the response of the model output to variations in the model input. In this paper, we consider the approximation of a function model $f(\boldsymbol{z}) : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ via a generalized Polynomial Chaos Expansion (PCE), which consists of a polynomial basis whose elements are orthogonal under the probability measure of the input variable $\boldsymbol{z}$ [27–29]. In particular, we focus on identifying the PCE coefficients from small number of function samples, which

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means that the number of samples is (severely) less than the cardinality of the linear approximation space.

Recently, based on the idea of compressive sensing [7, 10, 20], stochastic collocation method via ℓ_1-minimization techniques [11, 15, 17, 19, 21, 22, 30, 31] have been shown to be an efficient method to recover the sparse PCE coefficients from the underdetermined system. This is a natural approach to relax the original ℓ_0 minimization problem and trying to seek a sparse solution of the PCE vector.

Instead of ℓ_1-minimization, noticing the fact that $\lim_{q\to 0^+} \|\mathbf{c}\|_q^q = \|\mathbf{c}\|_0$, the ℓ_q minimization may provide a better approximation to the sparse solution. The advantage of this approach can be found in [18] and it has been widely used to recover sparse solutions [9, 13, 18]. The theoretical results in [25] give an interpretation to the intuitive observation that ℓ_q minimization provides a better approximation to ℓ_0 minimization than that ℓ_1 minimization can provide when the order of restricted isometry constants(RIC) is greater than 2k. Numerical comparisons between ℓ_1, ℓ_q minimization and other non-convex methods, such as reweighted ℓ_1 minimization, can be found in [13].

Based on the above observations, we will consider the stochastic collocation method via ℓ_q minimization to approximate the sparse PCE coefficients. The main contribution of this paper is that we establish some theroetical estimates for both exact recovery of sparse signals and the recoverability for general non-sparse signals. Following the works in [3], these estimates are obtained by using the key inequality between ℓ_2 and ℓ_q norm developed in [16]. The estimates are slightly different from the existing estimates of ℓ_q minimization in [13, 16, 25] and references therein. Although these estimates are not able to verify the improvement of the ℓ_q minimization over the ℓ_1 minimization in terms of the RIC bound δ_s, it is the foundation results to get the sparse recovery analysis for our bounded orthogonal system. We apply the ℓ_q minimization to the stochastic collocation method and obtain the sparse PCE coefficients for Legendre orthogonal polynomials. We present the recoverability results for both sparse polynomial functions and general non-sparse functions based on the new RIC bound estimates of the ℓ_q minimization. Various numerical results are presented to show that the ℓ_q minimization make a significant improvement over ℓ_1 minimization to achieve sparse solutions, not only in low dimensions but also in high dimensions.

The rest of the paper is organized as follows. In Section 2, we introduce the generalized polynomial chaos (gPC) approximation, set up the ℓ_1 minimization and ℓ_q minimization problems. Section 3 introduces some auxiliary results, presents the recoverability estimates for both sparse signals and non-sparse signals via the ℓ_q minimization approach, gives the theorems for recovering high-dimensional Legendre polynomial chaos. In Section 4, some numerical experiments are provided to explore the performance and accuracy of stochastic collocation via ℓ_q minimization comparing with ℓ_1 minimization and reweighted ℓ_1 minimization. We finally give some conclusions in Section 5.

2. Problem setup

Let $\boldsymbol{z} = (z^1, \cdots, z^d)^T$ be a set of random vectors defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with d mutually independent components, where each z^i takes values in $\Gamma^i \subset \mathbb{R}$.