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Itô-Taylor Schemes for Solving Mean-Field Stochastic Differential Equations

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Abstract. This paper is devoted to numerical methods for mean-field stochastic differential equations (MSDEs). We first develop the mean-field Itô formula and mean-field Itô-Taylor expansion. Then based on the new formula and expansion, we propose the Itô-Taylor schemes of strong order γ and weak order η for MSDEs, and theoretically obtain the convergence rate γ of the strong Itô-Taylor scheme, which can be seen as an extension of the well-known fundamental strong convergence theorem to the mean-field SDE setting. Finally some numerical examples are given to verify our theoretical results.

AMS subject classifications: 60H35, 65C30, 60H10
Key words: Itô-Taylor scheme, mean-field stochastic differential equation, mean-field Itô-Taylor formula, error estimate.

1. Introduction

Let $(\Omega,\mathscr{F},P;\{\mathscr{F}_t\}_{0\le t\le T})$ be a complete filtered probability space where $\{\mathscr{F}_t\}_{0\le t\le T}$ represents the natural filtration of a standard m-dimensional Brownian motion $W=\{W_t\}_{0\le t\le T}$. In this paper, we are interested in the numerical solutions of the following *mean-field stochastic differential equations* (MSDEs) on $(\Omega,\mathscr{F},P;\{\mathscr{F}_t\}_{0\le t\le T})$

$$dX_t=\mathbb{E}[b(t,X_t,\mu)]\Big|_{\mu=X_t}ds+\mathbb{E}[\sigma(t,X_t,\mu)]\Big|_{\mu=X_t}dW_t \tag{1.1}$$

with initial value X_{t_0}, for $0\le t_0<t\le T$, where t_0 and T are, respectively, the deterministic initial time and terminal time; the initial condition X_{t_0} is $\mathscr{F}_{t_0}$ measurable; $b:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^{d\times m}$ are the coefficient functions. It is worth noting that $b(\cdot,x',x)$ and $\sigma(\cdot,x',x)$ are $\mathscr{F}_t$-adapted for any fixed number x', x, and that the stochastic integrals with respect to W_s are of Itô-type.

Mean-field theory is developed to study the collective behaviors resulting from mutual interactions of individuals in various physical and sociological dynamical systems. Mean-field stochastic differential equations (MSDEs) and related problems have been extensively

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studied since 1960s, and have found many important applications in diverse areas such as kinetic gas theory [3], quantum mechanics [19], quantum chemistry [23], economics and finance [9], stochastic optimal control problems [21], McKean-Vlasov type partial differential equations (PDEs) [15, 17, 24, 26], and mean-field backward stochastic differential equations (MBSDEs) [4, 5]. As an important tool to understand properties or dynamics of the mean-field SDE systems, numerical solutions deserve special attention. Nevertheless, there are still few works on the numerical topic for MSDEs, in contrast to well-developed theory of numerical methods for standard SDEs (see, e.g., [11])

In this paper, after introducing the mean-field Itô formula and the mean-field Itô-Taylor expansion, we propose our Itô-Taylor schemes of strong order γ and weak order η for solving MSDEs. With $\gamma = 0.5, 1.0, 1.5$ and $\eta = 2.0$, the schemes become the Euler scheme, the Milstein scheme, the strong order 1.5, and the weak order 2.0 schemes, respectively. Error estimate of the strong order γ Itô-Taylor scheme is rigorously obtained, and our numerical results show that the schemes are effective, accurate and stable for solving MSDEs, and are consistent with our theoretical conclusions. This work is the basis of the study on numerical methods for mean-field *forward backward stochastic differential equations* (FBSDEs). In our near future, based on this work and our previous works [7, 8, 12, 27, 30–38] on solving FBSDEs, we will study numerical methods for solving mean-field FBSDEs with applications.

The rest of this paper is organized as follows. After introducing the mean-field Itô formula and the mean-field Itô-Taylor expansion in Section 2, we propose the strong and weak Itô-Taylor schemes for solving MSDEs in Section 3. In Section 4, the error estimates of the strong Itô-Taylor scheme are obtained. Numerical examples are carried out in Section 5, and some conclusions are given in Section 6.

2. The mean-field Itô formula and Itô-Taylor expansion

In this section, we will present the mean-field Itô formula and the mean-field Itô-Taylor expansion, which play fundamental roles in proposing our Itô-Taylor schemes for solving MSDEs in Section 3.

2.1. The solvability of MSDEs

We rewrite the MSDE (1.1) with initial value X_{t_0} in the following integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} \mathbb{E}\big[b(s,X_s,x)\big]\Big|_{x=X_s} ds + \int_{t_0}^{t} \mathbb{E}\big[\sigma(s,X_s,x)\big]\Big|_{x=X_s} dW_s, \tag{2.1}$$

for $t \in [t_0, T]$. The stochastic process X_t is called an L^2-adapted solution of the MSDE (2.1), if it is $\mathscr{F}_t$-adapted, square integrable, and satisfies (2.1). Here

$$L^p = \{X : \mathbb{E}[|X|^p] < \infty\}, \quad p \geq 1.$$

We set the following assumptions on the coefficients b and σ.