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# Mixed Fourier-Jacobi Spectral Method for Two-Dimensional Neumann Boundary Value Problems

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**Abstract.** In this paper, we propose a mixed Fourier-Jacobi spectral method for two dimensional Neumann boundary value problem. This method differs from the classical spectral method. The homogeneous Neumann boundary condition is satisfied exactly. Moreover, a tridiagonal matrix is employed, instead of the full stiffness matrix encountered in the classical variational formulation. For analyzing the numerical error, we establish the mixed Fourier-Jacobi orthogonal approximation. The convergence of proposed scheme is proved. Numerical results demonstrate the efficiency of this approach.

**AMS subject classifications**: 33C45, 65M70, 35J25

**Key words**: Mixed Fourier-Jacobi orthogonal approximation, spectral method, Neumann boundary value problem.

## 1. Introduction

In the past several decades, spectral method has become increasingly popular in scientific computing and engineering applications (cf. [4–8, 13] and the references therein). In most of these applications, one usually considers spectral methods for Dirichlet boundary value problems. However, it is also important to consider various problems with Neumann boundary condition. In a standard variational formulation, this kind of boundary condition is commonly imposed in a natural way. Unfortunately, this approach usually leads to a full stiffness matrix for approximating the second derivatives.

To overcome this disadvantage, Shen [12] first introduced a Legendre spectral method with essential imposition of Neumann boundary condition. Moreover, Auteri et al. [2] also studied the aforementioned spectral solver for the Neumann problem associated with

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Laplace and Helmholtz operators in rectangular domains. This method differs from the classical spectral methods for such problems, the homogeneous Neumann boundary condition is satisfied exactly for each basis. In particular, the proposed approach leads to a diagonal stiffness matrix, rather than a full matrix encountered in the classical variational formulation. Wang and Wang [18] analyzed the numerical errors of this algorithm. Meanwhile, Yu and Wang [19] also developed Jacobi spectral method with essential imposition of Neumann boundary condition for one-dimensional Neumann boundary value problems.

In this paper, we investigate two-dimensional Neumann boundary value problem, using the Fourier-Jacobi spectral method with essential imposition of Neumann boundary condition. The main advantage of such treatment consists in that: (i). the stiffness matrix is tridiagonal, in contrast to the full stiffness matrix encountered in the classical variational formulation; (ii). the conservation of certain physical quantities can be retained for time-dependent problems. It is pointed out that Wang and Guo [15] also dealt with a heat transfer inside a unit disc with Dirichlet boundary condition, using Fourier-Jacobi spectral method.

For analyzing the numerical error, we establish basic result on mixed Fourier-Jacobi orthogonal approximation, motivated by Guo and Wang [10, 11], and Wang and Guo [16, 17]. The convergence of proposed scheme is proved. We also present some numerical results to demonstrate the efficiency of this approach.

This paper is organized as follows. In the next section, we recall some properties and relevant results of Jacobi approximations. The mixed Fourier-Jacobi orthogonal approximation for Neumann problem are established in Section 3. In Section 4, we propose the mixed Fourier-Jacobi spectral method with essential imposition of Neumann boundary condition for a model problem and analyze its numerical error. In Section 5, we present some numerical results. The final section is for concluding remarks.

## 2. Preliminaries

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, and denote its inner product, semi-norm and norm by $(u, v)_{r,\chi,\Lambda}$, $|v|_{r,\chi,\Lambda}$ and $\|v\|_{r,\chi,\Lambda}$ respectively. In particular, $L^2_\chi(\Lambda) = H^0_\chi(\Lambda)$, $(u, v)_{\chi,\Lambda} = (u, v)_{0,\chi,\Lambda}$ and $\|v\|_{\chi,\Lambda} = \|v\|_{0,\chi,\Lambda}$. For any $r > 0$, we define the space $H^r_\chi(\Lambda)$ by space interpolation as in [3]. In cases where no confusion arises, $\chi$ may be dropped from the notations whenever $\chi(x) \equiv 1$.

For $\alpha, \beta > -1$, we denote by $J^{(\alpha,\beta)}_l(x)$ the Jacobi polynomial of degree $l$, which is the eigenfunction of the following Sturm-Liouville problem

$$\partial_x((1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x v(x)) + \lambda^{(\alpha,\beta)}_l (1-x)^\alpha(1+x)^\beta v(x) = 0, \qquad x \in \Lambda, \tag{2.1}$$

with the corresponding eigenvalue $\lambda^{(\alpha,\beta)}_l = l(l+\alpha+\beta+1)$, $l \geq 0$. The Jacobi polynomials fulfill the following recurrence relations (cf. [1, 9, 14]),

$$\partial_x J^{(\alpha,\beta)}_l(x) = \frac{1}{2}(l+\alpha+\beta+1)J^{(\alpha+1,\beta+1)}_{l-1}(x), \qquad l \geq 1, \tag{2.2}$$