Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. 7, No. 1, pp. 89-115

DOI: 10.4208/aamm.2013.m296
February 2015

Sensitivity Analysis and Computations of the Time Relaxation Model

Monika Neda[1], Fran Pahlevani[2,*] and Jiajia Waters[1]

[1] *Department of Mathematics, University of Nevada, Las Vegas, NV 89154, USA*
[2] *Department of Mathematics, Penn State University-Abington College, PA 19001, USA*

Received 20 July 2013; Accepted (in revised version) 6 August 2014

Abstract. This paper presents a numerical study of the sensitivity of a fluid model known as time relaxation model with respect to variations of the time relaxation coefficient χ. The sensitivity analysis of this model is utilized by the sensitivity equation method and uses the finite element method along with Crank Nicolson method in the fully discretization of the partial differential equations. We present a test case in support of the sensitivity convergence and also provide a numerical comparison between two different strategies of computing the sensitivity, sensitivity equation method and forward finite differences.

AMS subject classifications: 65M60, 65M12, 76D99

Key words: Sensitivity analysis, time relaxation model, Navier-Stokes equations, finite element method.

1 Introduction

Sensitivity investigations have become an important feature in understanding the fluid behavior. A meaningful solution for the Navier-Stokes equations at high Reynolds number requires computations with a fine mesh. This leads to expensive simulations regarding the storage of matrices and running time. Fluid models have been developed in order to avoid these obstacles. As it has been presented in [3], even when a fluid flow model has performed well in practice, the reliability of the approximated flow variables is often not addressed. If the model displays sensitivity to certain parameters, the resulting flow solution is not reliable. To that end, sensitivity analysis techniques provide a measure to compute solution uncertainties due to the variation of the selected parameter and determine a reliable interval for the parameter value. Over the years, there have been studies

*Corresponding author.
Email: monika.neda@unlv.edu (M. Neda), fxp10@psu.edu (F. Pahlevani), wangj16@unlv.nevada.edu (J. Waters)

on the sensitivity of fluid flows in different aspects, [3, 9, 23, 39, 41, 42]. Assessing model error that leads to uncertainty quantification is an important application of sensitivity analysis, see [35, 40] for recent works in this direction. Defining the sensitivity of state variables in a physical system as derivatives of state varaibles with respect to the selected parameter, there are basically two methods of numerically calculating the sensitivities: Forward Finite Difference method (FFD) and Sensitivity Equation Method (SEM). One simply uses finite differences and the other is based on forming an equation for the state variable sensitivity by differentiating the original model equation. In the latter approach, the resulting sensitivity equation is a linear equation and in most cases it is solved in tandem with the model equation when the state variables from the original system appear in the sensitivity equation. SEM is categorized by two different strategies: Continuous Sensitivity Equation Method (CSEM) and Automatic Differentiation Method (ADM). The difference between ADM and CSEM is in the order of operations of discretization and differentiation. CSEM implements differentiation first and then discretization, whereas ADM implements discretization first and then uses differentiation. There have been many works done using ADM, see [24, 25, 28, 29] for some examples. The possibility of combining these two methods is discussed in [12]. While finite difference quotient is easy to compute using a flow solver code, it might not be a reliable technique to compute sensitivites of a fluid model, see [7, 22]. In that aspect, the use of CSEM is preffered to that one of the finite difference, see [6, 39] for a camparison between these two methods in computing sensitivities. For computing flow sensitivity via CSEM, once the flow solver has converged only a linear solve is needed. This is computationally less expensive than running a code for calculating non-linear flow for two different parameter input in the attempt of calculating sensitivity via finite difference quotient. CSEM has been extensively used to compute the sensitivities with respect to different regularization parameters, see [5, 7, 8, 10, 19] for some examples among many others in the literature. This paper explores the sensitivity of a time relaxation type model with respect to a regularization parameter given below.

The governing equations of fluid motion are the Navier Stokes equations,

$$\begin{aligned} &\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p-\nu\Delta\mathbf{u}=\mathbf{f} && \text{in } \Omega\times[0,T],\\ &\nabla\cdot\mathbf{u}=0 && \text{in } \Omega\times[0,T], \end{aligned}$$

where $\mathbf{u}$ and p represent velocity vector and pressure respectively, ν represents the viscosity and $\mathbf{f}$ represents the body force. Time Relaxation model (TRM) was introduced by Stolz, Adams and Kleiser [17]. The model was computationally tested on compressible flows with shocks and on turbulent flows [1, 2, 17], i.e., on the aerodynamic noise [20]. A continuous finite element analysis for the model along with numerical results can be found in [17], while a discontinuous finite element analysis can be found in [36]. Preliminary sensitivity computations can be found in [37]. In [38] a computational study have been published of the Leray-α model with respect to the filter width. This model applies a regularization to the non-linear term in NSE in the form of $\overline{\mathbf{u}}\cdot\nabla\mathbf{u}$, where $\overline{\mathbf{u}}$ is calculated