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Accurate Simulation of Circular and Elliptic Cylindrical Invisibility Cloaks

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Abstract. The coordinate transformation offers a remarkable way to design cloaks that can steer electromagnetic fields so as to prevent waves from penetrating into the *cloaked region* (denoted by Ω_0, where the objects inside are invisible to observers outside). The ideal circular and elliptic cylindrical cloaked regions are blown up from a point and a line segment, respectively, so the transformed material parameters and the corresponding coefficients of the resulted equations are highly singular at the cloaking boundary $\partial\Omega_0$. The electric field or magnetic field is not continuous across $\partial\Omega_0$. The imposition of appropriate *cloaking boundary conditions* (CBCs) to achieve perfect concealment is a crucial but challenging issue.

Based upon the principle that a well-behaved electromagnetic field in the original space must be well-behaved in the transformed space as well, we obtain CBCs that intrinsically relate to the essential "pole" conditions of a singular transformation. We also find that for the elliptic cylindrical cloak, the CBCs should be imposed differently for the cosine-elliptic and sine-elliptic components of the decomposed fields. With these at our disposal, we can rigorously show that the governing equation in Ω_0 can be decoupled from the exterior region Ω_0^c, and the total fields in the cloaked region vanish under mild conditions. We emphasize that our proposal of CBCs is different from any existing ones.

Using the exact circular (resp., elliptic) Dirichlet-to-Neumann (DtN) non-reflecting boundary conditions to reduce the unbounded domain Ω_0^c to a bounded domain, we introduce an accurate and efficient Fourier-Legendre spectral-element method (FLSEM) (resp., Mathieu-Legendre spectral-element method (MLSEM)) to simulate the circular cylindrical cloak (resp., elliptic cylindrical cloak). We provide ample numerical results to demonstrate that the perfect concealment of waves can be achieved for the ideal circular/elliptic cylindrical cloaks under our proposed CBCs and accurate numerical solvers.

AMS subject classifications: 65Z05, 74J20, 78A40, 33E10, 35J05, 65M70, 65N35

Key words: Invisibility cloaks, singular coordinate transformations, cloaking boundary conditions, essential "pole" conditions, Mathieu functions, spectral-element methods, exact Dirichlet-to-Neumann (DtN) boundary conditions, perfect invisibility.

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1 Introduction

Since the groundbreaking works of Pendry, Schurig and Smith [50], and Leonhardt [33], transformation optics (or transformation electromagnetics) has emerged as an unprecedentedly powerful tool for metamaterial design (see [10, 15, 61] and many original references therein). The use of coordinate transformations has also been explored earlier by Greenleaf et al. [23] in the context of electrical impedance tomography. Perhaps, one of the most appealing applications of metamaterials is the invisibility cloak [34]. The mechanism of a cloak is typically based on a singular coordinate transformation of the Maxwell equations that can steer the electromagnetic waves without penetrating into the cloaked region, and thereby render the interior effectively "invisible" to the outside [50]. The first experimental demonstration of a two-dimensional cloak with a simplified model was realized by Schurig et al. [52], along with full-wave finite-element simulations [16, 67]. These impactive works have inspired a surge of developments and innovations (see [20, 29, 65] for an up-to-date review).

In this paper, we are largely concerned with mathematical and numerical study of the ideal circular cylindrical cloak using the transformation in Pendry et al. [50] and its important variant, i.e., the elliptic cylindrical cloak [13, 43]. The coordinate transformation in [50] suppresses a disk into an annulus so that the interior "empty" space constitutes the cloaked region (see Fig. 1). Such a "point-to-circle" blowup leads to new electric permittivity and magnetic permeability parameters, which are singular at the *inner boundary* (denoted by $r = R_1$) of the cloak. Accordingly, the coefficients of the governing equation are highly singular. The presence of singularities poses significant challenges for simulation, realization and analysis as well. A critical issue resides in *how to impose suitable conditions at the inner boundary, i.e., CBCs, to achieve perfect concealment of waves.* We highlight below some relevant studies and attempts, which are by no means comprehensive, given a large volume of existing literature.

- Ruan et al. [51] first analytically studied the sensitivity of the ideal cloak [50] to a small δ-perturbation of the inner boundary (i.e., from R_1 to $R_1+\delta$, while the material parameters remained unchanged) under the transverse-electric (TE) polarization. Their findings are (i) the ideal cloak in [50] is sensitive to a tiny perturbation of the boundary; (ii) the electric field is discontinuous across the inner boundary; and (iii) the perturbed cloak is nearly ideal in the sense that the magnitude of the fields penetrated into the cloaked region is small.

- Zhang et al. [66] provided deep insights into the physical effects, and found that the singular transformation gave rise to electromagnetic surface currents along the inner interface of the ideal cloak (also see [65]).

- To shield the incoming waves, the perfect magnetic conductor (PMC) condition (i.e., the tangential component of the magnetic field vanishes) was imposed at $r = R_1$ in finite-element simulations (see, e.g., [16, 35, 43]). Indeed, such a condition can be