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The Mass-Preserving S-DDM Scheme for Two-Dimensional Parabolic Equations

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Abstract. In the paper, we develop and analyze a new mass-preserving splitting domain decomposition method over multiple sub-domains for solving parabolic equations. The domain is divided into non-overlapping multi-bock sub-domains. On the interfaces of sub-domains, the interface fluxes are computed by the semi-implicit (explicit) flux scheme. The solutions and fluxes in the interiors of sub-domains are computed by the splitting one-dimensional implicit solution-flux coupled scheme. The important feature is that the proposed scheme is mass conservative over multiple non-overlapping sub-domains. Analyzing the mass-preserving S-DDM scheme is difficult over non-overlapping multi-block sub-domains due to the combination of the splitting technique and the domain decomposition at each time step. We prove theoretically that our scheme satisfies conservation of mass over multi-block non-overlapping sub-domains and it is unconditionally stable. We further prove the convergence and obtain the error estimate in L^2-norm. Numerical experiments confirm theoretical results.

AMS subject classifications: 65M06, 65M12, 65M55, 76S05

Key words: Parabolic equations, S-DDM, mass-preserving, unconditional stability, convergence, multi-blocks, non-overlapping

1 Introduction

Parabolic equations are widely used in science and engineering, which describe water head in groundwater modelling, pressure in petroleum reservoir simulation, and diffusion phenomena in heat propagation, etc (see, for example, [1–3]). Due to the computational complexity and the large computational cost in long term and large scale simulations, there are strong interests in developing efficient domain decomposition methods for solving these kinds time-dependent PDEs in high dimensions.

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Domain Decomposition Methods (DDMs) allow the reduction of the sizes of problems by dividing the large domain into smaller ones on which the PDEs can be solved by multiple computers in parallel, including the overlapping methods and the non-overlapping methods ([4,8,9,11,12,16,17], etc). Since non-overlapping methods have low computation and communication costs at each time step, the non-iterative explicit-implicit schemes on non-overlapping domain decompositions have been developed for solving large scale problems of parabolic types on massively parallel machine (see, [4–10, 13–15, 18, 19]). [4, 9] proposed the mixed/hybrid schemes, where the implicit schemes are used in sub-domains while the explicit schemes are used for the interface values on the interfaces of sub-domains. [6] further proposed the explicit-implicit methods for parabolic problems by either a multi-step explicit scheme or a high-order explicit scheme on the interfaces which relaxed the stability requirements. Further developments were done in [14, 19] by introducing the implicit correction step on the interfaces for improving the stability, where [14] analyzed theoretically the unconditional stability on the zigzag-shaped interfaces of sub-domains and obtained the convergence of the improved schemes in two dimensions. [18] studied the cell centered finite difference domain decomposition procedure for parabolic equations in one dimension. The fluxes at interface points were computed explicitly by nearby fluxes values at previous time. [15] proposed the parallel difference schemes for parabolic equations in two dimensions, where the predicted interface values on interfaces are employed some linear combination of the values on previous two time levels on the interfaces of sub-domains, which leads to a three level schemes. The methods in [4,6,18] can work efficiently for one dimensional problems and for stripe-divided sub-domains along one spatial variable for parabolic equations. More recently, over multiple block-divided sub-domains, [5, 10] developed an efficient splitting domain decomposition methods (S-DDM) by combining the splitting technique with the non-overlapping decomposition for solving parabolic equations in two dimensions and compressible contamination fluid flows in porous media. While the local multi-level schemes with local time steps are used to compute the interface values on the interfaces of sub-domains, the interior values in sub-domains are solved by the splitting one-dimensional implicit schemes. But, these previous methods [4–6, 9, 10, 14, 15, 19] do not satisfy the important physical law of mass conservation on multiple sub-domains. It is an important and challenging task to develop and analyze the mass-preserving domain decomposition methods over multi-block sub-domains.

In this paper, over multiple block-divided sub-domains, we propose and analyze a new mass-preserving splitting domain decomposition method for solving two-dimensional parabolic equations by combining the non-overlapping domain decomposition, the splitting technique and the coupling of the solution and its fluxes on staggered meshes. In the method, the global domain is divided into multi-block non-overlapping sub-domains. At each time step level, we take two stages to compute the solutions and fluxes on sub-domains. Firstly, the intermediate interface fluxes of x-direction are computed by a semi-implicit (explicit) flux scheme. The intermediate solutions and fluxes in the interiors of sub-domains are computed by a x-directional splitting implicit solution-