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A Simple Method for Computing Singular or Nearly Singular Integrals on Closed Surfaces

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Abstract. We present a simple, accurate method for computing singular or nearly singular integrals on a smooth, closed surface, such as layer potentials for harmonic functions evaluated at points on or near the surface. The integral is computed with a regularized kernel and corrections are added for regularization and discretization, which are found from analysis near the singular point. The surface integrals are computed from a new quadrature rule using surface points which project onto grid points in coordinate planes. The method does not require coordinate charts on the surface or special treatment of the singularity other than the corrections. The accuracy is about $O(h^3)$, where h is the spacing in the background grid, uniformly with respect to the point of evaluation, on or near the surface. Improved accuracy is obtained for points on the surface. The treecode of Duan and Krasny for Ewald summation is used to perform sums. Numerical examples are presented with a variety of surfaces.

AMS subject classifications: 65R20, 65D30, 31B10, 35J08

Key words: Boundary integral method, layer potential, nearly singular integrals, Laplace equation, surface integral, implicit surface.

1 Introduction

We present a simple, accurate method for computing singular or nearly singular integrals defined on a smooth, closed surface in three-space. This method can be used to evaluate single or double layer potentials for harmonic functions, or the velocity and pressure in Stokes flow due to forces on a surface. The point of evaluation could be on or near the

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surface. To evaluate the integral, the kernel is first replaced by a regularized version. A preliminary value is found using a new quadrature rule for surface integrals which has the advantage that it does not require coordinate systems or a triangulation on the surface. Instead we sum values of the integrand over quadrature points which project onto grid points in coordinate planes, in a way that would be high order accurate if the integrand were smooth. Corrections for the regularization and discretization are then added to achieve higher accuracy. These corrections are given by explicit formulas derived using asymptotic analysis near the singularity as in [4]. The resulting value of the integral has $\mathcal{O}(h^p)$ accuracy for $p<3$, uniformly for points of evaluation near the surface, where h is the grid spacing in $\mathbb{R}^3$. For points on the surface, the accuracy is significantly improved by using a special regularization; see Section 3.3. For efficient summation we use the treecode algorithm of Duan and Krasny [11] designed for kernels with Gaussian regularization. The method presented here could be used, for example, to find values of the potential at grid points in $\mathbb{R}^3$ close to the surface. It should be applicable to computations with moving surfaces for which good accuracy is needed without extensive work to represent the surface at each time step. Other kernels could be treated by the same approach, and the more accurate version of the method for computing values on the surface could be used for a variety of problems which can be formulated as integral equations.

The present approach is an improvement and extension of the grid-based boundary integral method of [4]. In the earlier work the integral was replaced by sums in coordinate charts using a partition of unity. The need for explicit coordinate systems requires knowledge of the surface that might be difficult to obtain for a moving surface. Furthermore, if the coordinate system is too distorted, the accuracy will be poor because the discretization error will fail to be controlled by the regularization. Here we avoid these disadvantages by using a more direct rule for computing surface integrals, which was introduced for smooth integrands in [30]. This quadrature rule uses projections on coordinate planes rather than coordinate charts. Given a rectangular grid in $\mathbb{R}^3$, quadrature points are chosen as those points on the surface which project onto grid points in the coordinate planes, for which the normal to the surface has direction away from the plane. Weights for the quadrature points are found from a partition of unity on the unit sphere, applied to the normal vector at the point. The weight functions on the sphere are chosen universally, and do not depend on the particular surface. The resulting quadrature rule for surface integrals has high order accuracy, as allowed by the smoothness of the integrand and the surface. In effect, the method uses the existence of coordinate patches without having to refer to them explicitly. The quadrature points can be found efficiently if, for example, the surface is given analytically or numerically as the level set of a function. Examples in [30] with smooth integrands illustrate the accuracy of this method with a variety of surfaces, including ones of large genus.

For a variety of problems in partial differential equations, solutions can be written as integrals over surfaces, using a known fundamental solution. These include harmonic functions, electromagnetic waves, and viscous fluid flow modeled by the Stokes equations. The specific representation makes this formulation an attractive approach for nu-