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Monotone Finite Volume Scheme for Three Dimensional Diffusion Equation on Tetrahedral Meshes

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Abstract. We construct a nonlinear monotone finite volume scheme for three-dimensional diffusion equation on tetrahedral meshes. Since it is crucial important to eliminate the vertex unknowns in the construction of the scheme, we present a new efficient eliminating method. The scheme has only cell-centered unknowns and can deal with discontinuous or tensor diffusion coefficient problems on distorted meshes rigorously. The numerical results illustrate that the resulting scheme can preserve positivity on distorted tetrahedral meshes, and also show that our scheme appears to be approximate second-order accuracy for solution.

AMS subject classifications: 65M06, 65M12, 65M55

Key words: Monotonicity, finite volume scheme, diffusion equation, tetrahedral meshes.

1 Introduction

The monotonicity of finite volume scheme is an important issue for accurately and efficiently solving diffusion equations and has been under research for a long time. In the context of anisotropic thermal conduction, the scheme without preserving monotonicity can cause heat to flow from regions of lower temperature to higher temperature, and it can result in negative values of temperature in regions of large temperature variations. The construction of monotone scheme has been an active field of research in recent years.

A new numerical schemes on distorted meshes should satisfy some desirable properties [1], including monotonicity, local conservation, linearity-preserving, stability, high accuracy, and simplicity. To our knowledge, there exists no linear scheme satisfying all the above properties. Usually, a scheme can possess some of the properties mentioned

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above. The classical multi-point flux approximation (MPFA) method [2,3] and nine-point scheme [4–6] are not monotone on general meshes. The schemes in [5,6] consider the case of diffusion coefficient being scalar only. The schemes in [7,8] are monotone under certain severe (geometric) restrictions. The restrictions on monotonicity of the MPFA methods are analyzed in [9–13].The sufficient condition to ensure the monotonicity of the mimetic finite difference is investigated in [14, 15]. Some algorithms in [16] based on slope limiters are proposed to preserve the monotonicity. In [17], based on repair technique and constrained optimization, two approaches have been suggested to enforce discrete extremum principle for linear finite element solutions of general elliptic equations with self-adjoint operator on triangular meshes.

The criteria for the monotonicity of control volume methods on quadrilateral meshes is derived in [10], which shows that it is impossible to construct linear nine-point methods which unconditionally satisfy the monotonicity criteria when the discretization satisfies local conservation and exact reproduction of linear solution.

A few nonlinear finite volume methods with monotonicity have been proposed in [18–22]. It is shown in [18] that the scheme is monotone only for parabolic equations and sufficiently small time steps. Some two-dimensional nonlinear finite volume schemes have been further developed and analyzed in [1], [19–22]. The scheme in [1] is monotone on triangular meshes for strongly anisotropic and heterogeneous full tensor coefficients. Based on an adaptive approach of choosing stencil in the construction of discrete normal flux, a nonlinear finite volume scheme for diffusion problems with anisotropic and heterogeneous full tensor coefficients on arbitrary star-shaped polygonal meshes is proposed in [19]. In [20] a nonlinear finite volume scheme satisfying the discrete extremum principle for diffusion equation on polygonal meshes is constructed. An interpolation-free nonlinear monotone scheme is presented in [21], and it has been extended to the advection diffusion equations on unstructured polygonal meshes in [22].

Up to now there are too many researches on discrete schemes for the two-dimensional diffusion problems. As for the three-dimensional case, there are also some finite volume methods on polyhedral meshes have been discussed, e.g., [23–28]. But monotonicity analysis for them has seldom been studied. An effective way to ensure the monotonicity property is to construct a numerical method such that the final discretization matrix is an M-matrix (see [29]). An M-matrix analysis for three-dimensional schemes, which is the extension of two-dimensional cases in [12, 13], has been discussed in [30, 31].

In [32] a nonlinear monotone scheme for 3D diffusion problems on unstructured tetrahedral meshes is proposed, which is a generalization of the scheme in [18] on 2D triangular meshes. To construct monotone schemes, the diffusion coefficient and the location of collocation point associated with the cell are restricted. Following the ideas of [19, 21], a nonlinear two-point flux approximation scheme is proposed in [33]. The important feature of this method is that most of auxiliary unknowns are interpolated from primary unknowns on the basis of a physical relationship. And it has been extended to the advection diffusion equations on unstructured polyhedral meshes in [34].

In this paper, we develop a nonlinear monotone scheme for three-dimensional dif-