布拉格天文钟的数学原理

The Mathematics Behind Prague's Horologe

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一、引言

欢迎来到捷克共和国的首都——布拉格。这个中欧城市不但拥有“千塔之城”的美誉，而且更被联合国教科文组织列入世界文化遗产。不少举世闻名的数学家、物理学家和天文学家都在这片土地上度过光辉的岁月，留下许许多多永不磨灭的历史印记。其中的代表人物包括文艺复兴时期的意大利哲学家乔尔达诺•布鲁诺(Giordano Bruno)、丹麦天文学家第谷•布拉赫 (Tycho Brahe)、其助手德国天文学家约翰尼斯•开普勒 (Johannes Kepler)、波希米亚数学家伯纳德•波尔查诺 (Bernard Bolzano)、法国数学家奥古斯汀•柯西 (August Cauchy)、挪威数学家尼尔斯•亨利克•阿贝尔 (Niels Henrik Abel)、奥地利数学及物理学家克里斯提昂•多普勒 (Christian Doppler)、奥地利物理学及哲学家恩斯特•马赫 (Ernst Mach)、相对论创立人德国理论物理学家阿尔伯特•爱因

1. Introduction

Welcome to Prague — the capital of the Czech Republic, called the “City of a Hundred Towers”, located in central Europe, and designated as a World Heritage site by UNESCO. Many famous mathematicians, physicists and astronomers have spent here very fruitful and creative years, and left unforgettable traces in Prague, in particular, Giordano Bruno, Tycho Brahe, Johannes Kepler, Bernard Bolzano, August Cauchy, Niels Henrik Abel, Christian Doppler, Ernst Mach, Albert Einstein and his mathematical colleague Georg Pick who was one of the people who taught him the tensor calculus. During their stays in Prague the above-mentioned scientists developed several fundamental mathematical and physical theories and engaged in related activities. For instance, in the beginning of the 17th century Kepler formulated the first two of his three laws of planetary motion based on Tycho Brahe’s observations. In the first half of the 19th century Bolzano constructed a nondifferentiable continuous function (of a fractal character) and wrote a treatise on infinite sets entitled Paradoxes of infinity (1851). In 1842 Doppler, professor of mathematics at the Prague Technical University, first lectured about his

斯坦 (Albert Einstein) 和他的大学同僚乔治•皮克 (Georg Pick)，而皮克是教授爱因斯坦张量微积分的数学家之一。上述科学家在布拉格生活期间，建立了几个对后世影响深远的数学和物理学理论，并且从事相关之研究工作。十七世纪初，开普勒根据第谷•布拉赫的观测结果，提出行星运动三大定律之第一、第二定律。十九世纪上半叶，波尔查诺给出了一个既有分形特征，又不可微分的连续函数，并且以无限数集 (infinite sets) 为题撰写了《无穷的诡论》(Paradoxes of Infinity, 1851)。1842年，布拉格理工大学数学教授多普勒在布拉格Ovocný trh(图1)的查理大学，首次就其创立的效应 (后来称为“多普勒效应”) 发表公开演讲。1911年至1912年，爱因斯坦在布拉格德国大学 (Prague German University) 担任理论物理学教授，醉心于广义相对论的研究工作。甚至著名捷克作家卡雷尔• 恰佩克 (Karel Capek) 也是在布拉格发明“robot”这个字 (捷克语Robota，意谓劳役、苦工)。本文后面会简单介绍这些与布拉格息息相关的伟人纪念碑和雕塑。接下来，本文集中讨论布拉格旧城广场中心的一个著名建筑物，分析其中有趣的数学问题。

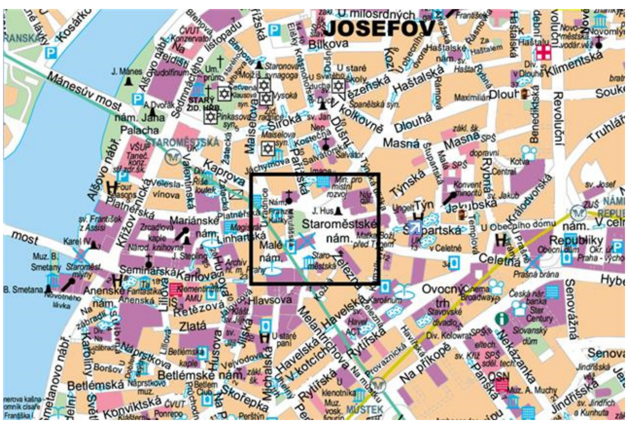

Fig.1 布拉格旧城区中心地图

Map of the Old Town in the center of Prague

布拉格旧城区中心(图1-2)有一个古色古香的天文钟(捷克语orloj，英语horologe)。无论是一般游客，或是热爱数学的人，都会慕名而来一睹这个举世稀有的珍品。本文将揭示天文钟与三角形数之间鲜为人知的关系，探讨三角形数的特性，以及这些特性如何提升大钟的准确度。

布拉格天文钟的数学模型设计来自简•安卓亚 (Joannes Andreae，捷克语Jan Ondrejuv，生于1375年，

effect (later called the Doppler effect) at Charles University at Ovocný trh (see Figure 1). Einstein, while a professor of theoretical physics at the Prague German University, worked on his theory of general relativity in 1911–1912. The famous Czech writer Karel Čapek invented the term “robot” in Prague. Before briefly detailing plaques, statues, and other memorials to these personalities of Prague at the end of this paper, we discuss some unexpected mathematics associated with a prominent building at the heart of the Old Town Square of Prague.

In the center of Old Town in Prague (see Figures 1 and 2), there is an astronomical clock (called “orloj” in the Czech language and horologe in English) — an interesting rarity often visited by many tourists, not only mathematical tourists. We found that there is a surprising connection between this clock and triangular numbers. In this article we take notice of special properties of these numbers that make the regulation of the bellworks more precise.

Fig 2. 旧城广场天文钟的位置

Location of the astronomical clock at the Old Town Square (Staroměstské náměstí)

The mathematical model of the astronomical clock of Prague was invented by Jan Ondřejův, called Šindel (Joannes Andreae, cca 1375–1456), a professor at Prague University founded in 1348 by the Emperor Charles IV. In 1410, Šindel was the rector there. The astronomical clock was realized by the skilled clockmaker Mikuláš (Nicholas) from Kadaň in 1410, i.e., exactly 600 years ago.

The astronomical clock of Prague is placed inside an almost 60 m high tower of the Old Town City Hall. The clock has two large dial-plates on the south wall of the tower (see Figure 3). Over the centuries the construction of the clock has been renovated several times, for example, by the clockmaker Jan from Růže (called Master Hanuš) around 1490. A memorial plaque devoted to the creators of the clock is on the left of the lower dial-plate.

The upper dial-plate of the astronomical clock is an astrolabe controlled by a clockwork mechanism. It represents a stereographic

卒于1456年)。安卓亚又名辛蒂尔 (Šindel)，在国王查理四世于1348年所创办的布拉格大学任教。1410年，辛蒂尔当上大学院长，天文钟的设计意念终于通过卡丹市 (Kadaň) 的钟匠密库拉斯 (捷克语Mikulás，即英语Nicholas) 得以实现。

布拉格天文钟设于旧城市政厅一座约六十米高的钟楼内，而两个大钟盘 (图3) 则镶嵌在钟楼南面的外墙上。六百年来，天文钟经历过几次大型翻新，其中一次约在1490年，由克伦洛夫城 (Ruze) 的钟表工匠简恩 (Jan，又名哈劳斯大师Master Hanus) 带领进行。天文钟下钟盘的左方特别设置一面纪念碑，用来表彰这些钟匠们的付出和贡献。

Fig. 4 天文钟的上钟盘
The upper dial-plate of the astronomical clock

Fig 3. 布拉格天文钟的两个钟盘
The dial-plates of the astronomical clock of Prague

projection of the celestial sphere from its North Pole onto the tangent plane passing through the South Pole. The center of the dial-plate thus corresponds to the South Pole of the celestial sphere. The smallest interior circle around the South Pole illustrates the Tropic of Capricorn, whereas the exterior circle illustrates the Tropic of Cancer. The concentric circle between them corresponds to the equator of the celestial sphere (see Fig 4).

An important property (known already to Ptolemy) of the stereographic projection is:

Any circle on the sphere which does not pass through the North Pole is mapped onto a circle as well.

Therefore, the ecliptic on the celestial sphere is projected on a circle, which is represented by the gilded ring with zodiac signs along the ecliptic. However, its center is not the South Pole, but the ring eccentrically rotates around this pole (see Figure 4). The astronomical clock also shows the approximate position of the Sun on the ecliptic, the motion of the Moon and its phases, and the rising, culmination and setting of the Sun, the Moon and zodiac signs.

The gilded solar hand indicates the Central-European time (CET) in the ring of Roman numerals. Note that the difference between CET and the original Prague local time is only 138 seconds. The clock-hand with a small gilded asterisk shows the sidereal time (see Figure 4). Twenty four golden Arabic

天文钟的上钟盘是一个靠发条机制控制而运作的星盘，象征天球 (celestial sphere) 由其北极通过南极落在切平面上的球极平面投影 (stereographic projection)。钟盘的中心点相当于天球的南极，南极四周的最小内圆代表南回归线，而外圆则代表北回归线，两者之间的同心圆相当于天球赤道 (图4)。

球极平面投影有一基本性质 (由古希腊天文学家托勒密Ptolemy提出)：

球体上所有异于北极的圆，经过球极平面投影法，在平面上的投影也是一个圆。

因此，天球黄道的投影也是圆形，用刻有十二星座图案的镀金钟圈表示。虽然天球黄道的中心点并不是南极点，但是镀金钟圈却神奇地绕着南极点转动(如图4所示)。此外，天文钟也指出了太阳在黄道上的大概位置、月球的运动和月相，以及日、月和十二星座各自的出、落和中天时间。

镀金太阳指针在罗马数字钟圈上转动，显示的是中欧时间 (Central European Time，简称CET)，值得留意的是，中欧时间和原来的布拉格当地时间相差只有138秒。旁边那枝镀金星星指针所显示的是恒星时(sidereal time)。最外那个钟圈上有金制的阿拉伯数字1至24，标示从日落起计算的古捷克时间；而下方黑色的阿拉伯数字1至12，是用来标示早在巴比伦时代已经开始使用的行星时间 (planetary hours)。行星时间则由日出开始算起，与古捷克时间的计算方法相反。

钟盘下方的黑色圆形部分代表天文曙暮光 (astronomical night)，即太阳处于地平线下18度的时段；外围的棕色部分象征黎明和黄昏 (AVRORA 和 CREPVSCVLV 标志着白昼和夜晚)，而 ORTVS 和 OCCASVS 则代表日出和日落。

天文钟的主装置有三个同心大齿轮，每个直径为116 厘米，最初由三个各有 24 齿的小齿轮驱动。第一个大齿轮有 365 齿，每个恒星日 (即 23 小时 56 分 4 秒) 推动星座钟圈转一周。第二个大齿轮有 366 齿，每一平太阳日 (mean sun day) 推动太阳指针转动一圈。由于地球环绕太阳的的公转轨道是椭圆形而非圆形，因此太阳在天球上的运动速度不均一。现时，星座钟圈的位置每年要经人手调校两次。第三个大齿轮有 379 齿，推动月

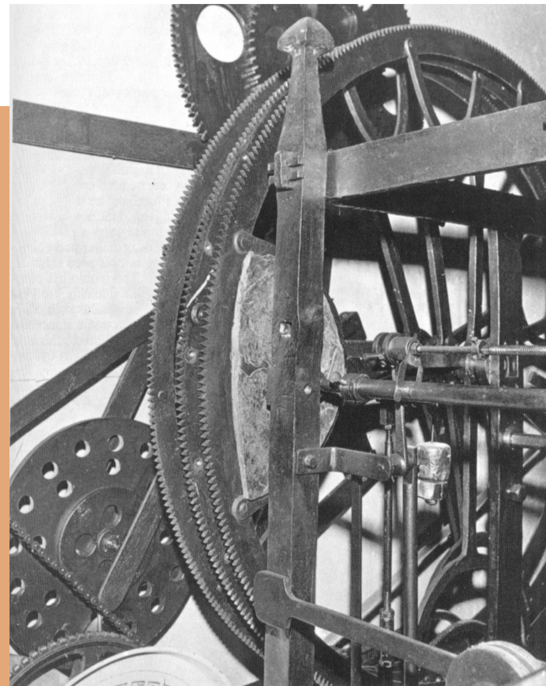

Fig. 5 主钟的详图(后面那三个同心大齿轮十五世纪初开始运作)
A detail of the main clock. The three large concentric gears behind are from the beginning of the 15th century

numerals are used for the ancient Czech time measured from sunset. Twelve black Arabic numerals denote planetary hours of the Babylonian time measured from sunrise.

The black circular area at the bottom of the dial-plate corresponds to the astronomical night, when the Sun is lower than 18° below horizon. The brown area stands for twilight (AVRORA in the morning and CREPVSCVLVM in the evening). Sunrise is denoted by ORTVS and sunset by OCCASVS.

In the main clockwork, there are three large original concentric gears of the same diameter 116 cm (see Figure 5) which were originally driven on one axis by three pinions, each with 24 teeth. The first gear has 365 teeth and turns round the zodiac ring once per sidereal day (23 hr 56 min 4 s). The second gear, which has 366 teeth, drives the solar pointer and turns round once per mean solar day. Since the true orbit of the Earth is elliptic, the Sun does not move uniformly on the celestial sphere. Therefore, the position of the zodiac ring is at present slightly corrected manually twice per year. The third gear, which has 379 teeth, drives the Moon's hand and rotates according to the mean apparent motion of the Moon. The lunar pointer is also at present manually corrected due to the elliptic orbit of the Moon. The lunar pointer (see Figure 4) is a hollow sphere with a hidden mechanism inside that displays the phases of the Moon. It was developed in the 17th century