Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2015.m1331

Vol. **8**, No. 4, pp. 475-495
November 2015

A Numerical Method for Solving Matrix Coefficient Heat Equations with Interfaces

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Received 02 November 2013; Accepted (in revised version) 04 July 2014

Abstract. In this paper, we propose a numerical method for solving the heat equations with interfaces. This method uses the non-traditional finite element method together with finite difference method to get solutions with second-order accuracy. It is capable of dealing with matrix coefficient involving time, and the interfaces under consideration are sharp-edged interfaces instead of smooth interfaces. Modified Euler Method is employed to ensure the accuracy in time. More than 1.5th order accuracy is observed for solution with singularity (second derivative blows up) on the sharp-edged interface corner. Extensive numerical experiments illustrate the feasibility of the method.

AMS subject classifications: 65N30

Key words: Heat equation, Sharp-edged interface, Jump condition, Matrix coefficient.

1. Introduction

Interface problems arise in problems involving multi-physics/multi-phase materials, which is common in fluid dynamics, electromagnetic, biological systems [1, 2], etc. Therefore, they attract considerable interest both theoretically and numerically in the past decades. However, due to the low global regularity and the irregular geometry of the interface, it is difficult to design a highly efficient method for such problems. In this paper, we seek the solution for heat equations with sharp-edged interface and matrix coefficient. The setting of the problem is as follows:

Consider a rectangular domain $\Omega = (x_{min}, x_{max}) \times (y_{min}, y_{max})$. Γ is an interface prescribed by the zero level-set $\{(x, y) \in \Omega \mid \phi(x, y) = 0\}$ of a level-set function $\phi(x, y)$.

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The unit normal vector of Γ is $\boldsymbol{n}=\frac{\nabla\phi}{|\nabla\phi|}$ pointing from $\Omega^-=\{(x,y)\in\Omega \mid \phi(x,y)\le 0\}$ to $\Omega^+=\{(x,y)\in\Omega \mid \phi(x,y)\ge 0\}$. Now the governing equation reads

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t}-\bigtriangledown\cdot(\beta(\boldsymbol{x},t)\bigtriangledown u(\boldsymbol{x},t))=f(\boldsymbol{x},t),\ \text{in } (\Omega\setminus\Gamma)\times(0,T), \tag{1.1}$$

in which $\boldsymbol{x}=(x_1,...,x_d)$ denotes the spatial variables and $\bigtriangledown$ is the gradient operator. The coefficient $\beta(\boldsymbol{x},t)$ is assumed to be a $d\times d$ matrix that is uniformly elliptic on each disjoint subdomain, Ω^- and Ω^+, and its components are continuously differentiable on each disjoint subdomain, but they may be discontinuous across the interface Γ. The right-hand side $f(\boldsymbol{x},t)$ is assumed to lie in $L^2(\Omega)$.

Given functions a and b along the interface Γ, we prescribe the jump conditions on $\Gamma\times(0,T)$

$$\begin{aligned}
&[u(\boldsymbol{x},t)]_\Gamma\equiv u^+(\boldsymbol{x},t)-u^-(\boldsymbol{x},t)=a(\boldsymbol{x},t),\\
&[(\beta(\boldsymbol{x},t)\bigtriangledown u(\boldsymbol{x},t))\cdot\boldsymbol{n}]_\Gamma\equiv\boldsymbol{n}\cdot(\beta^+(\boldsymbol{x},t)\bigtriangledown u^+(\boldsymbol{x},t))\\
&\qquad\qquad\qquad\qquad\qquad-\boldsymbol{n}\cdot(\beta^-(\boldsymbol{x},t)\bigtriangledown u^-(\boldsymbol{x},t))=b(\boldsymbol{x},t),
\end{aligned} \tag{1.2}$$

The superscripts "$\pm$" refer to limits taken from within the subdomains $\Omega^\pm$. Finally, we prescribe initial and boundary conditions

$$\begin{aligned}
&u(\boldsymbol{x},0)=u_0(\boldsymbol{x}),\ \text{in } \Omega,\\
&u(\boldsymbol{x},t)=g(\boldsymbol{x},t),\ \text{on } \partial\Omega\times(0,T).
\end{aligned} \tag{1.3}$$

for a given function g on the boundary $\partial\Omega$ and u_0 on the domain Ω.

Since the pioneer work of Peskin in 1977, much attention has been paid to solving interface problems. The Immersed Boundary Method [3,4] that he proposed uses a numerical approximation of the δ-function, which smears out the solution on a thin finite band around the interface Γ. In [5], the "immersed boundary" method was combined with the level set method, resulting in a first order numerical method that is simple to implement, even in multiple spatial dimensions. However, for both methods, the numerical smearing at the interface forces continuity of the solution at the interface, regardless of the interface condition $[u]=a$, where a might not be zero.

After that, in order to get higher order accuracy, researchers has done a lot of work on the finite difference solution for interface problems. The main idea is to use difference scheme and stencils carefully near the interface to incorporate jump conditions and achieve high order local truncation error using Taylor expansion. The Immersed Interface Method [6,7] incorporates the interface conditions into the finite difference scheme near the interface to achieve second-order accuracy based on a Taylor expansion in a local coordinate system. The corresponding linear system is sparse, but not symmetric or positive definite.

The Boundary Condition Capturing Method [8] uses the idea of Ghost Fluid Method [9] to capture the boundary conditions. The method extends the solution from one side across the interface using the jump conditions. The GFM is robust and simple to