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A Two-Grid Finite Element Method for Time-Dependent Incompressible Navier-Stokes Equations with Non-Smooth Initial Data

Deepjyoti Goswami[1,*] and Pedro D. Damázio[2]

[1] *Department of Mathematical Sciences, Tezpur University, Napaam, Sonitpur, Assam -784028, India.*
[2] *Department of Mathematics, Universidade Federal do Paraná, Centro Politécnico, Curitiba, Cx.P: 19081, CEP: 81531-990, PR, Brazil.*

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Abstract. We analyze here, a two-grid finite element method for the two dimensional time-dependent incompressible Navier-Stokes equations with non-smooth initial data. It involves solving the non-linear Navier-Stokes problem on a coarse grid of size H and solving a Stokes problem on a fine grid of size $h,\ h << H$. This method gives optimal convergence for velocity in H^1-norm and for pressure in L^2-norm. The analysis mainly focuses on the loss of regularity of the solution at $t = 0$ of the Navier-Stokes equations.

AMS subject classifications: 65M60; 65M12; 65M15

Key words: Two-grid, finite element, Navier-Stokes equations, non-smooth initial data, error estimates.

1. Introduction

We consider a two-grid semi-discrete finite element approximation to the two dimensional time-dependent incompressible Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}(x,t), \quad x \in \Omega,\ t > 0 \tag{1.1}$$

with incompressibility condition

$$\nabla\cdot\mathbf{u} = 0, \quad x \in \Omega,\ t > 0, \tag{1.2}$$

and initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \ \text{ in } \Omega, \quad \mathbf{u} = 0, \ \text{ on } \partial\Omega,\ t \geq 0. \tag{1.3}$$

*Corresponding author.

Here, Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\nu > 0$ is the viscosity. $\mathbf{u}$ and p are the velocity field and pressure, respectively. And $\mathbf{f}$ is a given force field.

Two-grid methods are well-established and efficient methods for solving non-linear partial differential equations. Due to high computational cost of solving a non-linear problem on a fine grid, we solve the original problem on a coarse grid and update the solution by solving a linearized problem on a fine grid. In other words, in the first step, we discretize the non-linear PDE on a coarse mesh, of mesh-size H and compute an approximate solution, say, $\mathbf{u}_H$. Then, in the second step, we solve a linear problem on a fine mesh, of mesh-size $h,\ h \ll H$, thereby, compute an approximate solution, say, $\mathbf{u}^h$. With appropriate h, H, we obtain comparable solution to that of Galerkin approximation on the fine grid, although, in this case, with far less computational cost, since, instead of solving a large non-linear system, we solve a small non-linear system and a large linear system.

In this article, we study the following two-grid finite element approximation for the problem (1.1)-(1.3): Compute a semi-discrete Galerkin finite element approximations $(\mathbf{u}_H, p_H)$, over a coarse mesh of mesh-size H. Then, refine this rough approximation $\mathbf{u}_H$ by solving the following Stokes problem:

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}_H \cdot \nabla\mathbf{u}_H + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \tag{1.4}$$

over a fine mesh of mesh-size $h \ll H$.

The above algorithm is nothing new and in fact, this and similar algorithms have been studied on numerous occasions, see [1,2], [5]- [7], [9]- [11], [13]- [15] and [18]- [25]. But to the best of our knowledge, no study has been done for non-smooth initial data. The importance of this, in the context of Navier-Stokes equations, has been pointed out by Heywood and Rannacher in their paper [16]. It has been noted that higher-order regularity estimates remain valid (as $t \to 0$), only in the presence of non-local compatibility conditions of various order, for the given data, at time $t = 0$. Since these conditions are not natural and are difficult to verify, we have to allow the possibility that regularity estimates are no longer bounded as $t \to 0$. These singularities are more prominent in case of non-smooth initial data. Since error analysis is based on various regularity assumptions, this is of vital importance. Hence the task at hand is to carry out the necessary analysis under realistically assumed data.

Two-grid method was first introduced by Xu [29, 30] for semi-linear elliptic problems and by Layton *et.al* [18–20] for steady Navier-Stokes equations. It was carried out for time dependent Navier-Stokes by Girault and Lions [11] for semi-discrete case. The method may vary depending on the algorithm, by formulating different linearized problem, to solve in the second step; as in the case of Navier-Stokes, one can chose a Stokes problem or an Oseen problem or a Newton step to solve on the fine mesh. Several works in this direction, involving both semi-discrete and fully discrete analysis, can be found in [1, 2, 5, 6, 9, 13–15, 21–23, 25] and references therein.

The two-grid methods are similar to non-linear Galerkin methods, post-processed and dynamical post-processed methods, in the sense that, all these method try to control the computational cost or efficiency by controlling the non-linear term, implement-