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Nonnegativity-Preserving Repair Techniques for the Finite Element Solutions of Degenerate Nonlinear Parabolic Problems

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Abstract. In the numerical simulation for nonlinear diffusion problems with degenerate diffusion coefficients, some classical methods are often invalid since they involve a harmonic mean of diffusion coefficients on the adjacent cells. To avoid such problem, we consider to use linear finite element method to solve a class of 1D degenerate nonlinear parabolic equations. This method can effectively capture the profile of true solution, but at the front it generates some nonphysical numerical oscillations, even brings forth negative values in numerical solution for approximating nonnegative physical quantities. In order to preserve the nonnegativity of true solution, we discuss three repair techniques for finite element solutions based on a posteriori corrections. The first one is a zero-setting method, in which we directly set those negative values to be zero. The second one is a local approach, in which any negative energy associate to some node is absorbed by the positive values around the current node. The third one is a global strategy, in which the total negative energy is redistributed to all positive values with a one-time effort. Numerical examples show that the numerical solution profile is improved remarkably by using the repair techniques for degenerate parabolic equations.

AMS subject classifications: 35K65, 65M60
Key words: Finite element method, repair technique, degenerate nonlinear parabolic equation.

1. Introduction

In this paper we shall discuss the finite element approximation of the following degenerate parabolic equation (DPE)

$$p_t-(K(p)p_x)_x = 0 \qquad \text{in } (a,b)\times(t_0,T], \tag{1.1a}$$

$$K(p(a,t))p_x(a,t)=K(p(b,t))p_x(b,t)= 0 \qquad \text{for } t\in\times(t_0,T], \tag{1.1b}$$

$$p(x,t_0)= g(x) \qquad \text{in } [a,b], \tag{1.1c}$$

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where $T < \infty$ and the diffusion coefficient $K(p)$ is a nonnegative function with $K(0) = 0$. Thus (1.1a) is a degenerate nonlinear parabolic equation. We assume the initial-value $g(x) \geq 0$ with a compact support. Then the set $\{p > 0\}$ has a free boundary $\Gamma(t)$. We make some further assumptions on $K(p)$ such that the free boundary has a finite speed of propagation. Note that the solution is not smooth across $\Gamma(t)$, since p has low regularity.

The system (1.1a)-(1.1c) arises in many areas of the physical sciences, including nonlinear heat and mass transfer, flow in porous media, combustion theory, incompressible fluid dynamics. The function $p(x,t)$ usually represents a concentration, or a temperature, or an energy, which should be nonnegative physically. Often the phenomenon of the finite speed of propagation is called the slow diffusion case. For the mathematical theories on degenerate nonlinear parabolic equations, we refer to [2, 3, 16, 31, 34] and the references cited therein.

Due to these interesting facts, numerical analysis on degenerate parabolic equations has been widely studied for several decades. Finite difference method applied to DPE has a long history in numerical simulation. In 1971, Graveleau and Jamet applied the finite different method to DPE in [14]. Tomoeda and Mimura in [30] investigated finite difference approximation to interface curves for porous medium equation (PME). Oberman [26] analyzed a convergent difference scheme for Hamilton-Jacobi equations and free boundary problems. Finite element method is another popular tool in the numerical computation for degenerate parabolic equations. Rose [27] developed an error estimate for linear finite element (for space)- backward Euler (for time) approximation for PME, which was improved by Nochetto and Verdi in [25]. A priori L^ρ error estimate for Galerkin approximation to fast diffusion equations was proved by Wei and Lefton [33]. Rulla and Walkington [28] considered fully discrete implicit schemes for the solution of the Stefan problem in two dimensions and proved error estimates. A local discontinuous Galerkin finite element method was applied to numerical simulation for 1-D porous medium equation by Zhang and Wu in [36]. The considered 1-D porous medium equation is

$$p_t - (p^m)_{xx} = 0, \tag{1.2}$$

which is a special case of (1.1a) with $K(p) = mp^{m-1}$ where m is a constant bigger than one. High order positivity preserving methods were developed by Liu *et al.* in [22], and by Guo and Yang in [15]. DiBenedetto and Hoff [6] proved that the finite difference solution of (1.2) converges to the correct solution, and that the numerical interface captures the true interface as well. Ebmeyer in [11] discussed the Galerkin approximation of a class of degenerate parabolic equations, and proved L^2-error estimates for the numerical solution. Duque *et al.* [8–10] applied a moving mesh method to PME with variable exponent in 2D with free boundaries, and studied the approximation and convergence of finite element method for porous medium equations. Ngo and Huang [24] presented an adaptive moving mesh finite element method for porous medium equation with and without variable exponents and absorption. In order to satisfy the physical nature of the porous medium equation, they proposed a nonnegativity preserving limiter. The fast diffusion equation and the porous medium equation were fully discretized by a Garlerkin scheme and the *a priori* error estimate was proved in [12, 13]. Recently, Droniou and Eymard in [7] applied