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Enlarging the Ball Convergence for the Modified Newton Method to Solve Equations with Solutions of Multiplicity under Weak Conditions

Ioannis K. Argyros[1,*] and Santhosh George[2]

[1] *1 Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA*
[2] *Department of Mathematical and Computational Sciences, NIT Karnataka, 575025, India*

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Abstract. The objective of this paper is to enlarge the ball of convergence and improve the error bounds of the modified Newton method for solving equations with solutions of multiplicity under weak conditions.

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1. Introduction

Many problems in applied sciences and also in engineering can be written in the form like

$$F(x)=0 \tag{1.1}$$

using mathematical modeling, where $F:\Omega\subseteq\mathscr{B}_1\longrightarrow\mathscr{B}_2$ is sufficiently many times differentiable and $\Omega,\mathscr{B}_1,\mathscr{B}_2$ are convex subsets in $\mathbb{R}$. In the present study, we pay attention to the case of a solution p of multiplicity $m>1$, namely, $F(p)=0, F^{(i)}(p)=0$ for $i=1,2,\cdots,m-1$, and $F^{(m)}(p)\neq 0$. The determination of solutions of multiplicity m is of great interest. In the study of electron trajectories, when the electron reaches a plate of zero speed, the function distance from the electron to the plate has a solution of multiplicity two. Multiplicity of solution appears in connection to Van Der Waals equation of state and other phenomena. The convergence order of iterative methods decreases if the equation has solutions of multiplicity m. Modifications in the iterative function are made to improve the order of convergence. The modified Newton's method (MN) defined for each $n=0,1,2,\cdots$ by

$$x_{n+1}=x_n-mF'(x_n)^{-1}F(x_n), \tag{1.2}$$

*Corresponding author. *Email addresses:* `iargyros@cameron.edu` (I. K. Argyros) and sgeorge@nitk.ac.in (S. George)

where $x_0 \in \Omega$ is an initial point is an alternative to Newton's method in the case of solutions with multiplicity m that converges with second order of convergence. A method with third order of convergence is defined by

$$x_{n+1} = x_n - \left(\frac{m+1}{2m}F'(x_n) - \frac{F''(x_n)F(x_n)}{2F'(x_n)}\right)^{-1} F(x_n). \tag{1.3}$$

Method (1.3) is an extension of the classical Halley's method of the third order. Another cubically convergence method was given by Traub [15]:

$$x_{n+1} = x_n - \frac{m(3-m)}{2}\frac{F(x_n)}{F'(x_n)} - \frac{m^2}{2}\frac{F(x_n)^2F''(x_n)}{F'(x_n)^3}. \tag{1.4}$$

Method (1.4) is an extension of the Chebyshev's method of the third order. Other iterative methods of high convergence order can be found in [5, 6, 9, 12, 15] and the references therein.

Let $B(p,\lambda) := \{x \in B_1 : |x-p| < \lambda\}$ denote an open ball and $\bar{B}(p,\lambda)$ denote its closure. It is said that $B(p,\lambda) \subseteq \Omega$ is a convergence ball for an iterative method, if the sequence generated by this iterative method converges to p, provided that the initial point $x_0 \in B(p,\lambda)$. But how close x_0 should be to p so that convergence can take place? Extending the ball of convergence is very important, since it shows the difficulty, we confront to pick initial points. It is desirable to be able to compute the largest convergence ball. This is usually depending on the iterative method and the conditions imposed on the function F and its derivatives. We can unify these conditions by expressing them as:

$$\left\|(F^{(m)}(p))^{-1}(F^{(m)}(x) - F^{(m)}(y)\right\| \leq \psi(\|x-y\|) \tag{1.5}$$

for all $x, y \in \Omega$, where $\psi : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous and nondecreasing function satisfying $\psi(0) = 0$. If we specialize function ψ, for $m \geq 1$ and

$$\psi(t) = \mu t^q, \ \ \mu > 0, \ \ q \in (0,1], \tag{1.6}$$

then, we obtain the conditions under which the preceding methods were studied in [4, 5, 12, 13, 16, 17]. However, there are cases where even (1.6) does not hold (see Example 4.1). Moreover, the smaller function ψ is chosen, the larger the radius of convergence becomes. The technique, we present next can be used for all preceding methods as well as in methods where $m = 1$. However, in the present study, we only use it for MN. This way, in particular, we extend the results in [4, 5, 12, 13, 16, 17]. In view of (1.5) there always exists a function $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing, satisfying

$$\left\|(F^{(m)}(p))^{-1}(F^{(m)}(x) - F^{(m)}(p)\right\| \leq \varphi_0(\|x-p\|) \tag{1.7}$$

for all $x \in \Omega$ and $\varphi_0(0) = 0$. We can always choose $\varphi_0(t) = \psi(t)$ for all $t \geq 0$. However, in general

$$\varphi_0(t) \leq \psi(t), \quad t \geq 0 \tag{1.8}$$