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OPTIMAL ORDER CONVERGENCE IMPLIES NUMERICAL SMOOTHNESS II: THE PULLBACK POLYNOMIAL CASE

SO–HSIANG CHOU AND TONG SUN

Abstract. A piecewise smooth numerical approximation should be in some sense as smooth as its target function in order to have the optimal order of approximation measured in Sobolev norms. In the context of discontinuous finite element approximation, that means the shape function needs to be numerically smooth in the interiors as well as across the interfaces of elements. In previous papers [2, 8] we defined the concept of numerical smoothness and stated the principle: numerical smoothness is necessary for optimal order convergence. We proved this principle for discontinuous piecewise polynomials on $\mathbb{R}^n, 1 \leq n \leq 3$. In this paper, we generalize it to include discontinuous piecewise non-polynomial functions, e.g., rational functions, on quadrilateral subdivisions whose pullbacks are polynomials such as bilinears, bicubics and so on.

Key words. Adaptive algorithm, discontinuous Galerkin, numerical smoothness, optimal order convergence.

1. Introduction

Consider the problem of approximating a function u defined on a domain Ω in $\mathbb{R}^n$ by a sequence of numerical solutions $\{u_h\}$ that are defined on subdivisions of Ω parametrized by the maximum mesh size h. The target function u may be the exact solution of a partial differential equation, and the sequence, discontinuous piecewise polynomials from a discontinuous Galerkin or finite volume method [6, 7], or post-processed finite element solutions to achieve superconvergence [11]. Now suppose that u is in $W_s^{p+1}(\Omega)$ (standard notation for Sobolev spaces here, supindex for the order of derivative and subindex for the L^s-based space) and that an optimal order approximation

$$||u - u_h||_{L^s(\Omega)} = \mathcal{O}(h^{p+1}), \tag{1}$$

holds, we would like to know what kind of smoothness u_h must have. For this purpose we defined across the interface smoothness in [2, 8] for $1 \leq n \leq 3$ and in particular for $n = 2$ it is as follows.

Definition 1.1. Interface Numerical Smoothness. *Let $\{\mathcal{Q}_h\}$ be a family of triangulations or quadriangulation (by quadrilaterals) of $\Omega \subset \mathbb{R}^n$. Let W_h be a function space such that*

$$W_h \subset \{v : \Omega \to \mathbb{R} : v|_\kappa \in C^{p+1}(\bar{\kappa}), \kappa \in \mathcal{Q}_h\},\ \ dim\ W_h < \infty.$$

Let $\{x_i\}_{i=1}^{N^\circ}$ be the set of all midpoints of interior edges. Then, $u_h \in W_h$ is said to be interface $W_s^{p+1}(\Omega)$-smooth, $s \geq 1$, if there is a constant $C_s > 0$, independent of h and u_h, such that

$$\sum_{i=1}^{N^\circ} h^2 \|D_i\|^s \leq C_s, \tag{2}$$

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and interface $W_\infty^{p+1}(\Omega)$-smooth, if there exists a constant $C_\infty > 0$, independent of h and u_h, such that

$$\max_{1\le i\le N^\circ} \|D_i\| \le C_\infty, \tag{3}$$

where the components of D_i are the scaled jumps $J_i^{(\alpha)}$ of partial derivatives at x_i

$$D_i^\alpha = J_i^{(\alpha)}/(h^{p+1-|\alpha|}), \quad J_i^{(\alpha)} := [\![\partial^\alpha u_h]\!]_{x_i}, \quad |\alpha| = k,\ 0 \le k \le p. \tag{4}$$

Two important examples of W_h are piecewise polynomial space and space of piecewise continuously differentiable functions whose pre-images under bilinear transformation are polynomial. It is most instructive just by looking at the $n = 1$ case, and see that several natural conditions for optimal convergence are already included, e.g., the scaled functional value $|D_i^0| \le C$ for all i in the case of $k = 0$, and at the other end in the case of $k = p$ that $|D_i^p| \le C$ or (2) with $s = 1$ implies the piecewise constant function $\frac{d^p u_h}{dx^p}$ has bounded variation, when W_h is the space of piecewise polynomials of degree at most p.

Intuitively, the smoothness of a numerical solution $u_h \in W_h$ should be measured by the boundedness of partial derivatives $\partial^\alpha u_h$. On an element $\kappa \in \mathcal{Q}_h$, by Taylor expansion around any point x_m in $\bar\kappa$, e.g., the center of κ or a point on the boundary of κ using one-sided derivatives, we see that boundedness of the quantities $\partial^\alpha u_h(x_m)$ would be sufficient to guarantee the interior smoothness, i.e., there exists a constant $M > 0$, independent of h, such that

$$|\partial^\alpha u_h(x_m)| \le M, \quad \forall\ |\alpha| = k,\ 0 \le k \le p. \tag{5}$$

On the other hand, intuitively the smoothness across the interface boundary of an element should be measured by the jumps of partials $J_i^{(\alpha)}$. The crucial part of Definition 1.1 is to point out one should use instead the scaled jump quantities D_i^α in (4). Notice that this definition does not involve any target solution u. Next, to give a corresponding interior numerical smoothness we replace the quantity D_i by F_i, the difference between the derivatives of u_h and the target u at x_m.

Definition 1.2. Interior Numerical Smoothness. *Let $u \in C^{p+1}(\Omega), \Omega \subset \mathbb{R}^2$ and let u_h be as in Def. 1.1 and let $\{x_i\}_{i=1}^{\mathcal{N}}$ be a collection of points $x_i \in \kappa_i \in \mathcal{Q}_h, 1 \le i \le \mathcal{N}$, where $\mathcal{N}$ is the number of elements in $\mathcal{Q}_h$. Then, u_h is said to be interior $W_s^{p+1}(\Omega)$-smooth, $s \ge 1$, if there is a constant C_s, independent of h and u_h, such that*

$$\sum_{i=1}^{\mathcal{N}} h^2 \|F_i\|^s \le C_s, \tag{6}$$

and interior $W_\infty^{p+1}(\Omega)$-smooth, if there exists a constant C_∞ independent of h and u_h such that

$$\max_{1\le i\le N^{\mathcal{T}}} \|F_i\| \le C_\infty, \tag{7}$$

where the components of F_i are the scaled differences between partial derivatives

$$F_i^\alpha = \partial^\alpha(u - u_h)(x_i)/(h^{p+1-|\alpha|}), \quad |\alpha| = k,\ 0 \le k \le p.$$

The main result that states optimal order convergence implies numerical smoothness is proved in Theorems 3.1 and 3.2. In particular, as a byproduct we have the following simultaneous approximation result: If

$$\|u - u_h\|_{L^\infty(\Omega)} \le Ch^{p+1}|u|_{W_\infty^{p+1}},$$