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A SECOND-ORDER CRANK-NICOLSON METHOD FOR TIME-FRACTIONAL PDES

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Abstract. Based on convolution quadrature in time and continuous piecewise linear finite element approximation in space, a Crank-Nicolson type method is proposed for solving a partial differential equation involving a fractional time derivative. The method achieves second-order convergence in time without being corrected at the initial steps. Optimal-order error estimates are derived under regularity assumptions on the source and initial data but without having to assume regularity of the solution.

Key words. Crank-Nicolson scheme, time-fractional equation, convolution quadrature, finite element method, error estimates.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, denote a convex polygonal/polyhedral domain with boundary $\partial\Omega$, and consider the problem

$$
\begin{cases} \partial_t u(x,t) - \Delta \partial_t^{1-\alpha} u(x,t) = f(x,t) & (x,t) \in \Omega \times \mathbb{R}_+, \\ \partial_t^{1-\alpha} u(x,t) = 0 & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = v(x) & x \in \Omega, \end{cases} \tag{1}
$$

where $f(x,t)$ denotes a given source function and $v(x)$ given initial condition. The operator $\Delta : D(\Delta) \to L^2(\Omega)$ denotes the Laplacian, defined on the domain $D(\Delta) = \{\phi \in H_0^1(\Omega) : \Delta\phi \in L^2(\Omega)\}$, and $\partial_t^{1-\alpha} u$ denotes the left-sided Caputo fractional time derivative of order $1 - \alpha \in (0, 1)$, defined by (c.f. [11, pp. 91])

$$
\partial_t^{1-\alpha} u(x,t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\partial u(x,s)}{\partial s} \mathrm{d}s, \tag{2}
$$

where $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} \mathrm{d}t$ denotes the Euler gamma function. We refer interested readers to [15, 21] for the regularity of solutions to (1) and its applications.

A number of numerical methods have been developed in the literature for solving PDE problems involving a fractional time derivative [3, 7, 12, 13, 14, 16, 19], among which the use of convolution quadrature (CQ) [12, 13] becomes more and more popular due to its excellent stability property and ease of implementation.

One of the main difficulties encountered when solving fractional evolution PDEs such as (1) is the low regularity of the solution in time (even with smooth initial data), which causes severe reduction of the convergence rates of high-order numerical schemes. In [3], Cuesta et al. overcame this difficulty by correcting the numerical scheme at the starting time step, which yielded second-order convergence of the numerical solutions based on certain regularity assumptions on the source and initial data. This idea was extended to the case $0 < \alpha < 1$ in [7] and [9], where second-order BDF and Crank-Nicolson type methods were proposed, respectively, for solving an equivalent formulation of (1). The schemes generally yield first-order

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convergence of the numerical solutions, but can be restored to second-order by correcting the schemes at several starting time steps. Of course, if a non-uniform mesh is used for time discretization, then the second-order convergence can be achieved without correction at the starting steps [16].

The models considered in [3, 7, 9, 16] are closely connected to (1), but they have different smoothing properties. As a result, the numerical schemes proposed in these previous works can not be applied directly to problem (1). In this paper, we develop a Crank-Nicolson scheme for problem (1) based on CQ in time and a continuous piecewise linear finite element method (FEM) in space. Inspired by [9], we combine the backward Euler CQ with a θ-type method for approximating $\Delta\partial_t^{1-\alpha}u$, and use the standard backward Euler method for approximating $\partial_t u$. Unlike [9], which approximates the equation at $t = t_n - \frac{\alpha\tau}{2}$, our method approximates the equation at $t = t_n - \frac{\tau}{2}$. The numerical method proposed in this paper is the only existing second-order method for (1) that does not require correction at the starting time steps.

For given initial data $v \in L^2(\Omega)$ and source $f \in W^{2,1}(0,T;L^2(\Omega))$, we prove the following error estimate:

$$\|u_h(t_n) - U_h^n\| \le C\tau^2\left(t_n^{-1}\|f(0)\| + \|f'(0)\| + \int_0^{t_n} \|f''(s)\|\mathrm{d}s\right), \tag{3}$$

where u_h and U_h^n denote the semidiscrete and fully discrete Galerkin finite element solutions, respectively. Here and below, for simplicity, we denote $u_h(t)$ and $f(t)$ by $u_h(x,t)$ and $f(x,t)$, respectively. The theoretical analysis is based on integral representations of u_h and U_h^n obtained by means of Laplace transform and generating function, a technique originating in [12, 13] and which proved to be powerful in [3, 8, 10, 9, 14, 17]. Numerical examples are presented to illustrate the convergence rate of the proposed method.

The rest of the paper is organized as follows. In Section 2, we present the fully discrete Crank-Nicolson Galerkin FEM for time-fractional PDE (1) and then state our main theoretical results. In Section 3, we prove optimal convergence rate for the approximate solution in time by using its integral representation and estimates of the resolvent operator. Numerical results are given in Section 4 to illustrate the theoretical analyses. Throughout this paper, we denote by C, with/without a subscript, a generic constant independent of h, n, and τ, which could be different at different occurrences.

2. The main results

In this section, we present the numerical method for approximating the solutions of (1) and state the main result of this paper.

2.1. Semidiscrete Galerkin FEM. We first only consider the case of discretization in space. 2

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of the domain Ω into d-dimensional simplexes, denoted by π_h, with a mesh size h $(0 < h < h_0)$. A continuous piecewise linear finite element space X_h over the triangulation $\mathcal{T}_h$ is defined by

$$X_h = \{\chi_h \in H_0^1(\Omega) : \chi_h|_{\pi_h} \text{ is a linear function}, \forall \pi_h \in \mathcal{T}_h\}.$$

Over the finite element space X_h, we define the L^2 projection $P_h : L^2(\Omega) \to X_h$ by

$$(P_h\varphi, \chi_h) = (\varphi, \chi_h) \quad \forall \chi_h \in X_h$$