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OPTIMAL ERROR ESTIMATES OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE TWO-DIMENSIONAL SINE-GORDON EQUATION ON CARTESIAN GRIDS

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Abstract. The sine-Gordon equation is one of the basic equations in modern nonlinear wave theory. It has applications in many areas of physics and mathematics. In this paper, we develop and analyze an energy-conserving local discontinuous Galerkin (LDG) method for the two-dimensional sine-Gordon nonlinear hyperbolic equation on Cartesian grids. We prove the energy conserving property, the L^2 stability, and optimal L^2 error estimates for the semi-discrete method. More precisely, we identify special numerical fluxes and a suitable projection of the initial conditions for the LDG scheme to achieve $p+1$ order of convergence for both the potential and its gradient in the L^2-norm, when tensor product polynomials of degree at most p are used. We present several numerical examples to validate the theoretical results. Our numerical examples show the sharpness of the $\mathcal{O}(h^{p+1})$ estimate.

Key words. Sine-Gordon equation, local discontinuous Galerkin method, energy conservation, L^2 stability, *a priori* error estimates, Cartesian grids.

1. Introduction

Developing energy-conserving and highly accurate numerical schemes to solve nonlinear hyperbolic partial differential equations (PDEs) is a very challenging scientific problem and is of fundamental importance to the simulation of waves and solitons. They are also important when dealing with coarse grids and large time steps. In this paper, we propose an energy-conserving local discontinuous Galerkin (LDG) method for the following two-dimensional sine-Gordon nonlinear hyperbolic equation

$$u_{tt} + \beta u_t + \sin(u) = \Delta u + f(x,y,t), \quad (x,y) \in \Omega = [a,b]\times[c,d],\ t \in [0,T], \tag{1a}$$

subject to the following initial conditions

$$u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = v_0(x,y), \quad (x,y) \in \Omega, \tag{1b}$$

and to either periodic boundary conditions

$$\begin{aligned} &u(x,c,t) = u(x,d,t), \quad u_y(x,c,t) = u_y(x,d,t),\\ &u(a,y,t) = u(b,y,t), \quad u_x(a,y,t) = u_x(b,y,t), \end{aligned} \tag{1c}$$

or mixed Dirichlet-Neumann boundary conditions

$$u = g_D, \quad (x,y) \in \partial\Omega_D \quad \text{and} \quad \nabla u \cdot \mathbf{n} = \mathbf{g}_N \cdot \mathbf{n}, \quad (x,y) \in \partial\Omega_N, \tag{1d}$$

for some given functions f, g, h, g_D, and $\mathbf{g}_N$. Here, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ is the boundary of the domain Ω, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, and $[0,T]$ is a finite time interval. The initial conditions $u_0(x,y)$ and $v_0(x,y)$ are the wave modes or kinks and velocity functions, respectively. In our theoretical analysis we select the initial/boundary conditions and the source, $f(x,y,t)$, such that the exact solution

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u is a smooth function on $\Omega \times [0, T]$. Our results may be extended to 3-D Cartesian meshes in straight forward manner. Details are not included to save space.

The nonlinear sine-Gordon equation plays an important role in modern physics; see *e.g.*, [44]. It is well-known that the sine-Gordon equation has soliton solutions in the one- and two-dimensional cases. The simplest of these solutions are called kinks and anti-kinks. It arises in many applications in physics, see *e.g.*, [25, 43, 53, 57, 61, 62]. For instance, the sine-Gordon equation arises in many different applications including propagation of magnetic flux on Josephson junctions, sound propagation in a crystal lattice, differential geometry, self-induced transparency, stability of fluid motion, laser physics, and particle physics. The two-dimensional equation (1a) arises in extended rectangular Josephson junctions, which consist of two layers of super conducting materials separated by an isolating barrier. The nonlinear term $\sin(u)$ is the Josephson current across an insulator between two superconductors [29]. Several other physical applications can be found in the review article by Barone *et al.* [25]. It is known (see *e.g.*, [64]) that the two-dimensional sine-Gordon equation (1a) with $\beta = f = 0$ and compactly supported or periodic boundary conditions admits the following important conserved quantity, called the total energy,

$$\begin{aligned} E(t) &= \frac{1}{2} \iint_\Omega \left(u_t^2 + u_x^2 + u_y^2 + 2(1 - \cos(u))\right) dxdy \\ &= E_k(t) + E_s(t) + E_p(t), \end{aligned} \tag{2}$$

where the kinetic, strain, and potential energies are, respectively, defined by

$$E_k(t) = \frac{1}{2} \iint_\Omega u_t^2 dxdy, \quad E_s(t) = \frac{1}{2} \iint_\Omega (u_x^2 + u_y^2) dxdy,$$

$$E_p(t) = \iint_\Omega (1 - \cos(u)) dxdy.$$

Several numerical schemes have been developed in the literature for the sine-Gordon equation [1, 2, 26, 27, 35, 39, 40, 42, 46, 47, 48, 50, 51, 52, 55, 58, 59, 63, 67, 68, 71]. Among these methods, the finite-difference method, the finite-element method, the pseudospectral technique, the domain decomposition method. Some of these schemes are known to be energy-conserving. For instance, some energy-conserving second-order finite difference schemes have been proposed in [27, 41, 49, 66] and the references therein. These schemes are designed by using central second differences to approximate the second derivative terms in the PDE. The difference among the finite difference schemes is in the discretization of the nonlinear term $\sin(u)$. However, to the best of the author's knowledge, the application of the LDG method to (1a) has not been considered in the literature.

The main motivation for the LDG method proposed in this paper originates from the LDG techniques which have been successfully applied to many PDEs arising from a wide range of applications. The LDG finite element method is an extension of the discontinuous Galerkin (DG) method aimed at solving PDEs containing higher than first-order spatial derivatives. It was first introduced by Cockburn and Shu [38] for solving convection-diffusion problems. Since then, several LDG schemes have been developed and analyzed for various higher-order ordinary and partial differential equations in one and multiple dimensions including nonlinear two-point boundary-value problems [20], convection-diffusion problems [3, 7, 10, 16, 24], second-order wave equations [6, 12, 13, 14, 17], the sine-Gordon