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A New Theoretical Estimate for the Convergence Rate of the Maximal Weighted Residual Kaczmarz Algorithm

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Abstract. In this note we prove a new theoretical estimate for the convergence rate of the maximal weighted residual Kaczmarz algorithm for solving a consistent linear system. The estimate depends only on quantities that are easy to compute and not on the number of equations in the system. We compare the maximal weighted residual Kaczmarz algorithm and the greedy randomized Kaczmarz algorithm by two sets of examples. Numerical results show that the maximal weighted residual Kaczmarz algorithm requires almost the same number of iterations as that of the greedy randomized Kaczmarz algorithm for underdetermined linear systems and less iterations for overdetermined linear systems. Due to less computational cost in the row index selection strategy, the maximal weighted residual Kaczmarz algorithm is more efficient than the greedy randomized Kaczmarz algorithm.

AMS subject classifications: 65F10, 65F20
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1. Introduction

The Kaczmarz method [15], proposed by the Polish mathematician Stefan Kaczmarz, is a popular iterative method for solving consistent linear systems. Due to its simplicity, the Kaczmarz method has found applications in many fields, such as computer tomography [12, 16, 22], image reconstruction [13, 25], digital signal processing [3, 19], etc.

The convergence of the Kaczmarz method is not hard to show. However, useful theoretical estimates for the convergence rate are difficult to obtain. The difficulty is well reflected by the fact that the convergence rate of the method depends strongly on the row index selection strategy. Known estimates for the convergence rate of the deterministic Kaczmarz

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method either depend on the number of equations in the system (which makes the corresponding estimate very bad for extremely overdetermined systems; see, e.g., [1,21]) or are based on quantities of the coefficient matrix that are hard to compute and difficult to compare with convergence rates of other iterative methods (see, e.g., [5,6,8] and the references therein).

Numerical experiments show that using the rows of the coefficient matrix in the Kaczmarz method in random order, rather than in their given order, can often greatly improve the convergence [14,22]. In an influential paper [26], Strohmer and Vershynin proposed a randomized Kaczmarz method which converges linearly in expectation. The convergence rate depends only on the scaled condition number of the coefficient matrix. This result was extended and refined in various directions including inconsistent, underdetermined, or rank-deficient linear systems; see, e.g., [7,9–11,17,18,20,23,24,27].

Recently, Bai and Wu [2] proposed a new randomized row index selection strategy, which is aimed at grasping larger entries of the residual vector at each iteration, and constructed a greedy randomized Kaczmarz algorithm. They proved that the convergence of the greedy randomized Kaczmarz algorithm is faster than that of the randomized Kaczmarz algorithm [26]. Inspired by their proof, we prove a new theoretical estimate for the convergence rate of the maximal weighted residual Kaczmarz algorithm [21], which depends only on quantities of the coefficient matrix that are easy to compute and not on the number of equations in the system. We provide two sets of examples to compare the performance of the maximal weighted residual Kaczmarz algorithm and the greedy randomized Kaczmarz algorithm. Numerical results show that the maximal weighted residual Kaczmarz algorithm requires almost the same number of iterations as that of the greedy randomized Kaczmarz algorithm for underdetermined linear systems and less iterations for overdetermined linear systems. In all cases, the maximal weighted residual Kaczmarz algorithm spends much less computing time.

Organization of the note. In the rest of this section, we give some notation. In Section 2, we review the maximal weighted residual Kaczmarz algorithm, the randomized Kaczmarz algorithm, and the greedy randomized Kaczmarz algorithm. In Section 3, we prove the new theoretical estimate for the convergence rate of the maximal weighted residual Kaczmarz algorithm. Numerical examples are given in Section 4. We present brief concluding remarks in Section 5.

Notation. For any random variable ξ, let $\mathbb{E}\left[\xi\right]$ denote its expectation. Throughout the paper all vectors are assumed to be column vectors. For any vector $\mathbf{u} \in \mathbb{R}^m$, we use $\mathbf{u}^{\mathrm{T}}$, u_i, and $\|\mathbf{u}\|_2$ to denote the transpose, the ith entry, and the Euclidean norm of $\mathbf{u}$, respectively. For any matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, we use $\mathbf{A}^{\mathrm{T}}$, $\mathbf{A}^{\dagger}$, $\|\mathbf{A}\|_{\mathrm{F}}$, range($\mathbf{A}$), and $\sigma_r(\mathbf{A})$ to denote the transpose, the Moore-Penrose generalized inverse, the Frobenius norm, the column space, and the smallest nonzero singular value of $\mathbf{A}$, respectively. We denote the rows of $\mathbf{A}$ by $\{\mathbf{a}_i^{\mathrm{T}}\}_{i=1}^m$. That is to say,

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}.$$