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High Order Finite Difference Scheme based on DG Boundary Treatment (FDbDG)

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Abstract. Due to its computational efficiency, high-order finite difference (FD) method is attractive, but the difficulty of treating boundary hampers the practical application in complex flow simulation. In this work, we propose a novel high-order FD scheme based on discontinuous Galerkin (DG) boundary treatment (FDbDG) where a DG method based on variational principle is applied to provide the flow properties in the vicinity of the boundary with desirable derivative information in time. In order to carefully combine the finite element and finite difference, Hermite weighted essentially non-oscillatory (HWENO) interpolation is adopted to build the HWENO flux for interior FD scheme and HWENO reconstruction is used to construct the degrees of freedom in the DG flux for boundary variational method. Several typical test cases are selected to evaluate the treatment for FD boundary. Numerical results show the proposed FDbDG method can reach arbitrary order of accuracy including boundary region with non-essentially oscillations.

AMS subject classifications: 35Q30, 65M60, 76N15

Key words: Finite difference scheme, discontinuous Galerkin method, Hermite weighted essentially non-oscillatory schemes, boundary treatment.

1 Introduction

Recently, high-order numerical methods (third-order and above) with low numerical diffusion and dispersion errors have been widely developed and applied to resolve complex fluid structures in a variety of applications. Finite difference (FD) type schemes, adopted to resolve the conservation laws in differential form with one degree of freedom (DOF) in

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each cell, are generally considered as highly efficient and easily achieve high-order accuracy. Furthermore, FD type schemes designed on simpler uniform structured mesh can obtain high parallel efficiency on a large-scale computer. Above features make high-order FD method still very attractive.

Unfortunately, a high-order FD regime suffers several major challenges in resolving conservation laws on complex geometry.

Interpolation of high-order accuracy suffers oscillations near a discontinuity called Gibbs phenomena. A lot of research work has been done to eliminate oscillations. The Essentially Non-Oscillatory (ENO) idea was firstly proposed [1] to enhance stability. In [2–4], the Weighted ENO (WENO) schemes was developed, using a convex combination of all candidate stencils instead of just one as in the original ENO, to obtain higher order accuracy. In [5–7], a more compact Hermite WENO (HWENO) idea was proposed and employed in FD type schemes. The HWENO schemes adopted smaller stencils to obtain high-order accuracy. Nonlinear weighted compact (WCNS) methods based on compact high-order nonlinear schemes were introduced in [8–10] and also showed good performance for discontinuity capture.

Lack of abundant derivative information near the boundary causes difficulty in obtaining uniform high-order accuracy for problems involving complex geometry. There have been many successful numerical methods to address the challenge. One indirect approach is the immersed boundary (IB) method, which was introduced in [11]. IB method used a forcing function on physical boundaries to make a modification of origin partial differential equations. The method had been widely used to solve incompressible flows and fluid-structure problems on moving geometries. One can refer to the survey article [12] for more details. Usually, a direct approach is to obtain the values at several ghost points near the boundary by extrapolation. In fact, the key is that the boundary conditions imposed on physical boundaries are short of abundant derivative information to obtain high-order accuracy. A second order accurate embedded boundary method for the wave equation was developed in [13–15]. The idea was extended to solve conservation laws where slope limiters were adopted to avoid oscillations in [16]. Of course, for outflow boundary, the extrapolation can meet the upwind and physical principles, and the WENO extrapolation may be adopted to acquire higher order accuracy with enhanced stability. However, it can cause instability when used on an inflow boundary, solid wall boundary, etc., where the more abundant information should be imposed on physical boundary. Huang *et al.* developed a Lax-Wendroff (LW) type boundary condition for a third-order finite difference method in [17]. Recently, an inverse Lax-Wendroff (ILW) technique was developed by using the idea for time dependent problems in [18]. The main idea is to convert time derivatives and tangential derivatives to normal derivatives by repeatedly using the PDEs on the inflow boundary and to get the derivative information and derive ghost point values by Taylor expansion. WENO type extrapolation was adopted to handle the strong discontinuities to prevent the undershoot and overshoot. However, the algebra of the ILW procedure was very heavy for two-dimensional