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AN OPTIMAL EDG METHOD FOR DISTRIBUTED CONTROL OF CONVECTION DIFFUSION PDES

XIAO ZHANG, YANGWEN ZHANG, AND JOHN R. SINGLER

Abstract. We propose an embedded discontinuous Galerkin (EDG) method to approximate the solution of a distributed control problem governed by convection diffusion PDEs, and obtain optimal a priori error estimates for the state, dual state, their fluxes, and the control. Moreover, we prove the optimize-then-discretize (OD) and discrtize-then-optimize (DO) approaches coincide. Numerical results confirm our theoretical results.

Key words. Distributed optimal control, convection diffusion, embedded discontinuous Galerkin method, error analysis, optimize-then-discretize, discrtize-then-optimize.

1. Introduction

We study the following distributed optimal control problem:

$$\min J(u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Omega)}, \quad \gamma > 0, \tag{1}$$

subject to

$$\begin{aligned} -\Delta y + \boldsymbol{\beta}\cdot\nabla y &= f + u \quad \text{in } \Omega,\\ y &= g \qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ is a Lipschitz polyhedral domain with boundary $\Gamma = \partial\Omega$, $f \in L^2(\Omega)$, $g \in C^0(\partial\Omega)$, and the vector field $\boldsymbol{\beta}$ satisfies

$$\nabla\cdot\boldsymbol{\beta} \leq 0. \tag{3}$$

Optimal control problems for convection diffusion equations have been extensively studied using many different finite element methods, such as standard finite elements [11–13], mixed finite elements [13, 35, 39], discontinuous Galerkin (DG) methods [16, 21, 33, 34, 36, 40, 41] and hybrid discontinuous Galerkin (HDG) methods [17, 18]. HDG methods were first introduced by Cockburn et al. in [4] for second order elliptic problems, and they have subsequently been applied to many other problems [2, 3, 5, 7, 8, 23–26, 32]. HDG methods keep the advantages of DG methods, but have a lower number of globally coupled degrees of freedom compared to mixed methods and DG methods. However, the degrees of freedom for HDG methods is still larger compared to standard finite element methods. Embedded discontinuous Galerkin (EDG) methods were first proposed in [15], and then analyzed in [6]. EDG methods are obtained from the HDG methods by forcing the numerical trace space to be continuous. This simple change significantly reduces the number of degrees of freedom and make EDG methods competitive for flow problems [27] and many other applications [9, 10, 19, 27, 29].

In [38], we utilized an EDG method for a distributed optimal control problem for the Poisson equation. We obtained optimal convergence rates for the state,

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dual state and the control, but *suboptimal* convergence rates for their fluxes. This suboptimal flux convergence rate for the Poisson equation is a limitation of the EDG method with equal order polynomial degrees for all variables [6]. However, Zhang, Xie, and Zhang recently proposed a new EDG method and proved optimal convergence rates for all variables for the Poisson equation [37]. This new EDG method is obtained by simply using a lower degree finite element space for the flux. In this work, we use this new EDG method to approximate the solution of the above convection diffusion distributed optimal control problem, and in Section 3 we prove optimal convergence rates for all variables.

There are two main approaches to compute the numerical solution of PDE constrained optimal control problems: the optimize-then-discretize (OD) and discretize-then-optimize (DO) approaches. In the OD approach, one first derives the first-order necessary optimality conditions, then discretizes the optimality system, and then solves the resulting discrete system by utilizing efficient iterative solvers [31]. In the DO approach, one first discretizes the PDE optimization problem to obtain a finite dimensional optimization problem, which is then solved by existing optimization algorithms, such as [1,28]. The discretization methods for which these two approaches coincide are called *commutative*. Intuitively, the DO approach is more straightforward in practice; however, not all discretization schemes are commutative. In the non-commutative case, the DO approach may result in badly behaved numerical results; see, e.g., [20, 22]. Therefore, devising commutative numerical methods is very important. In Section 2, we prove the EDG method studied here is commutative for the convection diffusion distributed control problem. Moreover, we provide numerical examples to confirm our theoretical results in Section 4.

2. EDG scheme for the optimal control problem

2.1. Notation. Throughout the paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. Specifically, $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$. We denote the L^2-inner products on $L^2(\Omega)$ and $L^2(\Gamma)$ by

$$
\begin{aligned}
(v, w) &= \int_\Omega vw \quad \forall v, w \in L^2(\Omega),\\
\langle v, w\rangle &= \int_\Gamma vw \quad \forall v, w \in L^2(\Gamma).
\end{aligned}
$$

Define the space $H(\text{div}, \Omega)$ as

$$H(\text{div}, \Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^d, \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\}.$$

Let $\mathcal{T}_h$ be a collection of disjoint elements that partition Ω. We denote by $\partial\mathcal{T}_h$ the set $\{\partial K : K \in \mathcal{T}_h\}$. For an element K of the collection $\mathcal{T}_h$, let $e = \partial K \cap \Gamma$ denote the boundary face of K if the $d-1$ Lebesgue measure of e is non-zero. For two elements K^+ and K^- of the collection $\mathcal{T}_h$, let $e = \partial K^+ \cap \partial K^-$ denote the interior face between K^+ and K^- if the $d-1$ Lebesgue measure of e is non-zero. Let ε_h^o and ε_h^∂ denote the set of interior and boundary faces, respectively. We denote by ε_h the union of ε_h^o and ε_h^∂. We finally introduce

$$(w, v)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (w, v)_K, \qquad \langle\zeta, \rho\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \langle\zeta, \rho\rangle_{\partial K}.$$