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Existence of Solutions to Elliptic Equation with Exponential Nonlinearities and Singular Term

XUE Yimin* and CHEN Shouting

School of Mathematics and Physics, Xuzhou University of Technology, Xuzhou 221018, China.

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Abstract. In this paper, we consider an elliptic equation with exponential nonlinearities and singular term. By constructing the corresponding variational framework, and using a Singular Trudinger-Moser inequality due to Li, Mountain-pass theorem and the Ekeland's variational principle, we get a nontrivial positive weak solution.

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Key Words: Singular Trudinger-Moser inequality; Mountain-pass theorem; exponential growth.

1 Introduction and main results

Motivated by Adimurthi and Yang [1], Yang [2], in this paper, we concerned the elliptic differential equation

$$-\mathrm{div}\left(|\nabla u|^{N-2}\nabla u\right)+V(x)|u|^{N-2}u-\alpha\|u\|^{N-p}_{L^p(\mathbb{R}^N)}|u|^{p-2}u=\frac{f(x,u)}{|x|^{\beta}}, \tag{1.1}$$

where $x\in\mathbb{R}^N, N\geq 2$ is an integer, $V(x)\in C(\mathbb{R}^N,\mathbb{R})$ denotes all continuous functions from $\mathbb{R}^N$ to $\mathbb{R}$, $p\geq N$, $0\leq\beta<N$, $f(x,u)$ has exponential growth like $e^{\alpha u^{N/(N-1)}}$ as $|u|\to\infty$ and

$$0\leq\alpha<\lambda_{N,p}:=\inf_{u\in E,u\not\equiv 0}\frac{\int_{\mathbb{R}^N}\left(|\nabla u|^N+V_0|u|^N\right)\mathrm{d}x}{\left(\int_{\mathbb{R}^N}\left(|u|^p\right)\mathrm{d}x\right)^{N/p}}, \tag{1.2}$$

V_0 will be determined later. More details can be founded in [3–7].

*Corresponding author. *Email addresses:* `xueym@xzit.edu.cn` (Y. M. Xue), `stchen1980@163.com` (S. T. Chen)

Let E be the function space defined by

$$E=\left\{u\in W^{1,N}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)|u|^N\mathrm{d}x<\infty\right\}.$$

For convenience, we define a function $\zeta:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$

$$\zeta(N,s)=e^s-\sum_{k=0}^{N-2}\frac{s^k}{k!}=\sum_{k=N-1}^{\infty}\frac{s^k}{k!},\quad N\geq 2. \tag{1.3}$$

In the following, We list conditions on $V(x)$ and $f(x,s)$ in order to obtain our results.

(H_1) $V(x)\geq V_0>0$ in $\mathbb{R}^N$ for some $V_0>0$;

(H_2) $\frac{1}{V(x)}\in L^{\frac{1}{N-1}}(\mathbb{R}^N)$;

(H_3) There exist positive real constants α_0,a_1,a_2 such that

$$|f(x,s)|\leq a_1 s^{N-1}+a_2\zeta\left(N,\alpha_0 s^{\frac{N}{N-1}}\right),\ \forall(x,s)\in\mathbb{R}^N\times\mathbb{R}^+;$$

(H_4) There exist $\mu>N$ such that

$$0<\mu F(x,s)\equiv\mu\int_0^s f(x,t)\mathrm{d}t\leq sf(x,s);$$

(H_5) There exist positive real constants R_0,M_0 such that

$$F(x,s)\leq M_0 f(x,s),\qquad \forall x\in\mathbb{R}^N,s\geq R_0.$$

According to [2], we assume throughout this paper

$$f(x,s)\equiv 0,\qquad \forall(x,s)\in\mathbb{R}^N\times(-\infty,0). \tag{1.4}$$

It follows from (H_1) that E is a reflexive Banach space endowed the norm

$$\|u\|_{E,\alpha}=\left(\int_{\mathbb{R}^N}\left(|\nabla u|^N+V(x)|u|^N\right)\mathrm{d}x-\alpha\|u\|^N_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{N}}. \tag{1.5}$$

Now, we define a singular eigenvalue of the N-Laplace operator, for $\forall\ 0\leq\beta<N$

$$\lambda_\beta=\inf_{u\in E,u\neq 0}\frac{\|u\|_{E,\alpha}}{\int_{\mathbb{R}^N}\frac{|u|^N}{|x|^\beta}\mathrm{d}x}. \tag{1.6}$$