Commun. Comput. Phys.
doi: 10.4208/cicp.2019.js60.03

Vol. **26**, No. 5, pp. 1307-1334
November 2019

Long Time Error Analysis of Finite Difference Time Domain Methods for the Nonlinear Klein-Gordon Equation with Weak Nonlinearity

Weizhu Bao[1], Yue Feng[1,*] and Wenfan Yi[2]

[1] *Department of Mathematics, National University of Singapore, Singapore 119076.*
[2] *College of Mathematics and Econometrics, Hunan University, Changsha, 410082, Hunan Province, P.R. China.*

Received 20 February 2019; Accepted (in revised version) 12 April 2019

Abstract. We establish error bounds of the finite difference time domain (FDTD) methods for the long time dynamics of the nonlinear Klein-Gordon equation (NKGE) with a cubic nonlinearity, while the nonlinearity strength is characterized by ε^2 with $0<\varepsilon\leq 1$ a dimensionless parameter. When $0<\varepsilon\ll 1$, it is in the weak nonlinearity regime and the problem is equivalent to the NKGE with small initial data, while the amplitude of the initial data (and the solution) is at $\mathcal{O}(\varepsilon)$. Four different FDTD methods are adapted to discretize the problem and rigorous error bounds of the FDTD methods are established for the long time dynamics, i.e. error bounds are valid up to the time at $\mathcal{O}(1/\varepsilon^\beta)$ with $0\leq\beta\leq 2$, by using the energy method and the techniques of either the cut-off of the nonlinearity or the mathematical induction to bound the numerical approximate solutions. In the error bounds, we pay particular attention to how error bounds depend explicitly on the mesh size h and time step τ as well as the small parameter $\varepsilon\in(0,1]$, especially in the weak nonlinearity regime when $0<\varepsilon\ll 1$. Our error bounds indicate that, in order to get "correct" numerical solutions up to the time at $\mathcal{O}(1/\varepsilon^\beta)$, the ε-scalability (or meshing strategy) of the FDTD methods should be taken as: $h=\mathcal{O}(\varepsilon^{\beta/2})$ and $\tau=\mathcal{O}(\varepsilon^{\beta/2})$. As a by-product, our results can indicate error bounds and ε-scalability of the FDTD methods for the discretization of an oscillatory NKGE which is obtained from the case of weak nonlinearity by a rescaling in time, while its solution propagates waves with wavelength at $\mathcal{O}(1)$ in space and $\mathcal{O}(\varepsilon^\beta)$ in time. Extensive numerical results are reported to confirm our error bounds and to demonstrate that they are sharp.

AMS subject classifications: 35L70, 65M06, 65M12, 65M15, 81-08

Key words: Nonlinear Klein-Gordon equation, finite difference time domain methods, long time error analysis, weak nonlinearity, oscillatory nonlinear Klein-Gordon equation.

*Corresponding author. *Email addresses:* `matbaowz@nus.edu.sg` (W. Bao), `fengyue@u.nus.edu` (Y. Feng), `wfyi@hnu.edu.cn` (W. Yi)

1 Introduction

Consider the nonlinear Klein-Gordon equation (NKGE) with a cubic nonlinearity on a torus $\mathbb{T}^d$ $(d=1,2,3)$ [23, 27, 36, 37] as

$$\begin{aligned}
&\partial_{tt}u(\mathbf{x},t)-\Delta u(\mathbf{x},t)+u(\mathbf{x},t)+\varepsilon^2 u^3(\mathbf{x},t)=0, \quad \mathbf{x}\in\mathbb{T}^d, \quad t>0,\\
&u(\mathbf{x},0)=\phi(\mathbf{x}), \quad \partial_t u(\mathbf{x},0)=\gamma(\mathbf{x}), \quad \mathbf{x}\in\mathbb{T}^d.
\end{aligned} \tag{1.1}$$

Here t is time, $\mathbf{x}\in\mathbb{R}^{\mathbf{d}}$ is the spatial coordinates, $u:=u(\mathbf{x},t)$ is a real-valued scalar field, $0<\varepsilon\le 1$ is a dimensionless parameter, and $\phi(\mathbf{x})$ and $\gamma(\mathbf{x})$ are two given real-valued functions which are independent of ε. The NKGE is a relativistic (and nonlinear) version of the Schrödinger equation and it is widely used in quantum electrodynamics, particle and/or plasma physics to describe the dynamics of a spinless particle in some extra potential [4, 7, 13, 22, 33, 34, 36]. Provided that $u(\cdot,t)\in H^1(\mathbb{T}^d)$ and $\partial_t u(\cdot,t)\in L^2(\mathbb{T}^d)$, the NKGE (1.1) is time symmetric or time reversible and conserves the *energy* [5, 19], i.e.,

$$\begin{aligned}
E(t):=&\int_{\mathbb{T}^d}\left[|\partial_t u(\mathbf{x},t)|^2+|\nabla u(\mathbf{x},t)|^2+|u(\mathbf{x},t)|^2+\frac{\varepsilon^2}{2}|u(\mathbf{x},t)|^4\right]d\mathbf{x}\\
\equiv&\int_{\mathbb{T}^d}\left[|\gamma(\mathbf{x})|^2+|\nabla\phi(\mathbf{x})|^2+|\phi(\mathbf{x})|^2+\frac{\varepsilon^2}{2}|\phi(\mathbf{x})|^4\right]d\mathbf{x}:=E(0)=\mathcal{O}(1), \quad t\ge 0.
\end{aligned} \tag{1.2}$$

We remark here that, when $0<\varepsilon\ll 1$, rescaling the amplitude of the wave function u by introducing $w(\mathbf{x},t)=\varepsilon u(\mathbf{x},t)$, then the NKGE (1.1) with weak nonlinearity can be reformulated as the following NKGE with small initial data, while the amplitude of the initial data (and the solution) is at $\mathcal{O}(\varepsilon)$:

$$\begin{aligned}
&\partial_{tt}w(\mathbf{x},t)-\Delta w(\mathbf{x},t)+w(\mathbf{x},t)+w^3(\mathbf{x},t)=0, \quad \mathbf{x}\in\mathbb{T}^d, \quad t>0,\\
&w(\mathbf{x},0)=\varepsilon\phi(\mathbf{x}), \quad \partial_t w(\mathbf{x},0)=\varepsilon\gamma(\mathbf{x}), \quad \mathbf{x}\in\mathbb{T}^d.
\end{aligned} \tag{1.3}$$

Again, the above NKGE (1.3) is time symmetric or time reversible and conserves the *energy* [5, 19], i.e.,

$$\begin{aligned}
\bar{E}(t):=&\int_{\mathbb{T}^d}\left[|\partial_t w(\mathbf{x},t)|^2+|\nabla w(\mathbf{x},t)|^2+|w(\mathbf{x},t)|^2+\frac{1}{2}|w(\mathbf{x},t)|^4\right]d\mathbf{x}=\varepsilon^2 E(t)\\
\equiv&\int_{\mathbb{T}^d}\left[\varepsilon^2|\gamma(\mathbf{x})|^2+\varepsilon^2|\nabla\phi(\mathbf{x})|^2+\varepsilon^2|\phi(\mathbf{x})|^2+\frac{\varepsilon^4}{2}|\phi(\mathbf{x})|^4\right]d\mathbf{x}:=\bar{E}(0)=\mathcal{O}(\varepsilon^2).
\end{aligned} \tag{1.4}$$

In other words, the NKGE with weak nonlinearity and $\mathcal{O}(1)$ initial data, i.e. (1.1), is equivalent to it with small initial data and $\mathcal{O}(1)$ nonlinearity, i.e. (1.3). In the following, we only present numerical methods and their error bounds for the NKGE with weak nonlinearity. Extensions of the numerical methods and their error bounds to the NKGE with small initial data are straightforward.