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O-convexity of Orlicz-Bochner Spaces with Orlicz Norm

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Abstract: In this paper we give some characterizations of O-convexity of Banach spaces, and show the criteria for O-convexity in Orlicz-Bochner function space $L_M(\mu, X)$ and Orlicz-Bochner sequence space $l_M(X_s)$ endowed with Orlicz norm. Moreover, we give a sufficient condition for the dual of such a space to have the fixed point property.

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1 Introduction

It is well known that convexities and reflexivity play an important role in Banach space theory. Since B-convexity (write (BC)) was introduced by Beck[1], some revelent properties including uniform non-squareness (write (U-NS)) and P-convexity (write (PC)) were given by Brown[2], Giesy[3], James[4] and Kottman[5]. From their achievements we know that

$$\text{(UC) or (US)} \Rightarrow \text{(U-NS)} \Rightarrow \text{(Rfx) and (BC)},$$

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where (UC) denotes uniform convexity, (US) uniform smoothness, and (Rfx) reflexivity. A natural and interesting question raised by Brown[6]: "Is there a B-convex space that is not P-convex?" Though Giesy[7] and James[8] provided answer to the above question, Naidu and Sastry[9] introduced a new geometric conception in a Banach space X named O-convexity (write (OC)): if there exists an $\varepsilon > 0$ and an $n_0 \in \mathbf{N}^+$ such that for every

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$x^{(1)}, x^{(2)}, \cdots, x^{(n_0)} \in S(X)$ there holds

$$\min\{\|x^{(i)} - x^{(j)}\|, \|x^{(i)} + x^{(j)}\| : i, j \le n_0, i \ne j\} \le 2(1 - \varepsilon).$$

In [9], the authors showed

$$\text{(U-NS) or (PC)} \Rightarrow \text{(OC)} \Rightarrow \text{(S-Rfx)} \Rightarrow \text{(BC)},$$

where (S-Rfx) denotes super-reflexivity. They also proved that $X \bigoplus Y$ normed by $\|(x, y)\| = (\|x\|^p + \|y\|^p)^{\frac{1}{p}}$ is O-convex whenever X and Y are O-convex for $1 \le p < \infty$. In recent years, many works indicate that O-convexity is closely related to the fixed point property (see [10] and [11]).

For Orlicz-Bochner function space $L_M(\mu, X)$ or Orlicz-Bochner sequence space $l_M(X_s)$ with Orlicz norm, a fundamental question is that whether or not a geometrical property lifts from X to $L_M(\mu, X)$, or X_s to $l_M(X_s)$. The answer may often be guessed, but usually, the result exceed the guess and the proof is nontrivial. Various kinds of convexity for Lebesgue-Bochner space $L_p(\mu, X)$, Orlicz-Bochner function space $L_M(\mu, X)$ and Orlicz-Bochner sequence space $l_M(X_s)$ were carried out by many authors (see [12]–[17], etc).

In this paper, a characterization of O-convexity of $L_M(\mu, X)$ or $l_M(X_s)$ endowed with Orlicz norm is given. As a corollary of the main result we get that for $1 < p < \infty$, the equi-O-convexity of $\{X_s\}$ implies the O-convexity of $l_p(X_s)$, and the O-convexity of X implies the O-convexity of $L_p(\mu, X)$. Moreover, we show that the dual space of $L_M(\mu, X)$ (or $l_M(X_s)$) has the fixed point property whenever L_M and X are O-convex (or l_M and X_s are equi-O-convex).

Let $(X_s, \|\cdot\|_s)$ be Banach spaces and $S(X_s)$ be the unit sphere of the space X_s. Let $\mathbf{N}^+$, $\mathbf{R}$ and $\mathbf{R}^+$ denote the set of positive natural numbers, reals and positive reals, respectively. A function M is called an Orlicz function if $M: \mathbf{R} \to \mathbf{R}^+$ is even, convex, $M(u) = 0$ if and only if $u = 0$, $\lim\limits_{u\to 0} u^{-1}M(u) = 0$ and $\lim\limits_{u\to\infty} u^{-1}M(u) = \infty$. Define a modular of $x = \{x(s)\}_{s=1}^{\infty}$ (where every $x(s) \in X_s$) by $\rho_M(x) = \sum\limits_{i=1}^{\infty} M(\|x(s)\|_s)$. By Orlicz-Bochner sequence space $l_M(X_s)$ we mean the linear space

$$l_M(X_s) = \{x = (x(1), x(2), \cdots) : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

equipped with Orlicz norm

$$\|x\|_M = \inf_{k>0} \frac{1}{k}[1 + \rho_M(kx)].$$

Suppose that (Ω, Σ, μ) is a non-atomic Lebesgue measure space. For an X-valued measurable function $u(t)$, we call $\rho_M(u) = \int_{\Omega} M(\|u(t)\|)\mathrm{d}\mu$ the modular of u. Similarly as above we have the Orlicz-Bochner function space

$$L_M(\mu, X) = \{x = x(t) : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\}$$

endowed with Orlicz norm.

In this paper, L_M means $L_M(\mu, \mathbf{R})$, and so for l_M. For every Orlicz function M we define its complementary function $N: \mathbf{R} \to [0, \infty)$ by the formula

$$N(v) = \sup_{u>0}\{u|v| - M(u)\}, \qquad v \in \mathbf{R}.$$