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Nonlinear Parabolic Equations with Singular Coefficient with Respect to the Unknown and with Diffuse Measure Data

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Abstract. An existence and uniqueness result of a renormalized solution for a class of doubly nonlinear parabolic equations with singular coefficient with respect to the unknown and with diffuse measure data is established. A comparison result is also proved for such solutions.

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1 Introduction

In the present paper we establish an existence and uniqueness result of a renormalized solution for a class of nonlinear parabolic equations of the type

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \mathrm{div}(\mathcal{A}(x,u)\nabla u) + \lambda\, u = \mu & \text{in } Q, \\ b(u(t=0)) = b(u_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

In Problem (1.1) the framework is the following: Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$, T is a positive real number and $Q = \Omega \times (0,T)$ with the lateral boundary $\partial\Omega \times (0,T)$ and $u_0 \in L^1(\Omega)$, $u_0 \leq m$ a.e. in Ω, where m is a positive real number. The function b is assumed

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to be a $\mathcal{C}^1$-function defined on $\mathbb{R}$, such that which is strictly increasing. The Carathéodory function $\mathcal{A}$ defined on $\Omega\times]-\infty,m[$, satisfies

$$\lim_{s\to m^-}\mathcal{A}(x,s)=+\infty \quad \text{for a.e. } x\in\Omega.$$

The first difficulty in solving this equation is defining the field $\mathcal{A}(x,u)\nabla u$ on the subset $\{(x,t);\, u(x,t)=m\}$ of Q, since, on this set, $\mathcal{A}(x,u)=+\infty$.

The second difficulty is represented here by the presence of the measure data μ and the nonlinear term $b(u)$. To overcome these difficulties, we use in this paper the framework of renormalized solutions. A large number of papers was then devoted to the study of renormalized (or entropy) solution of parabolic problems with rough data under various assumptions and in different contexts: in addition to the references already mentioned see, [1–12].

The existence and uniqueness of a renormalized solution of (1.1) have been proved in [13] (see also [14]) in the stationary case where $\mathcal{A}(x,u)=d(u)+A(u)$ with $d(r)=(d_i(r))_{1\le i\le N}$ is a diagonal matrix defined on an interval $]-\infty,m[$ such that there exists an index $p\in\{1,\cdots,N\}$ such that $\lim_{r\to m^-}d_p(r)=+\infty$, where the matrix $A(r)\in\mathcal{C}^0(\mathbb{R},\mathbb{R}^{N\times N})$ and where $\mu\in L^2(\Omega)$ (see also [15,16]).

In the stationary and evolution cases of $u_t-\mathrm{div}(A(x,t,u)\nabla u)=f$ in Q, where the matrix $A(x,t,s)$ blows up (uniformly with respect to (x,t)) as $s\to m^-$ and where $f\in L^1(Q)$, the existence of renormalized solution has been proved in [2].

We call a finite measure μ diffuse if it does not charge sets of zero $2-$*capacity* (see Section 2 for the definition) and $\mathcal{M}_0(Q)$ will denote the subspace of all diffuse measures in Q. According to a representation theorem for diffuse measures proved in [17], for every $\mu\in\mathcal{M}_0(Q)$, there exist $f\in L^1(Q)$, $g\in L^2(0,T;H_0^1(\Omega))$ and $\chi\in L^2(0,T;H^{-1}(\Omega))$ such that $\mu=f+\chi+g_t$ in $\mathcal{D}'(Q)$.

In the case of

$$u_t-\sum_{i=1}^N\frac{\partial}{\partial x_i}\Big(d_i(u)\frac{\partial u}{\partial x_i}\Big)=\mu,$$

where the coefficients $d_i(s)$ are continuous on an interval $]-\infty,m[$ such that there exists an index $p\in\{1,\cdots,N\}$ such that $\lim_{r\to m^-}d_p(r)=+\infty$, the data $u_0\in L^1(\Omega)$ such that $u_0\le m$, and $\mu\in\mathcal{M}_0(Q)$, the existence of renormalized solution has been proved in [18].

When b is assumed to satisfy $0<b_0\le b'(r)\le b_1$, $\forall r\in\mathbb{R}$ and $\mathcal{A}(x,u)\nabla u$ is replaced by $a(x,t,\nabla u)$, and $\mu\in\mathcal{M}_0(Q)$, the existence and uniqueness of renormalized solution have been established in [19].

In the case of

$$u_t-\mathrm{div}(a(t,x,u,\nabla u))+\mathrm{div}(\Phi(u))=f+\mathrm{div}g \quad \text{in } Q,$$

where Φ is a continuous function, $f\in L^1(Q)$ and $g\in(L^{p'}(Q))^N$ the existence of renormalized solution has been proved in [20].