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General Energy Decay of Solutions for a Wave Equation with Nonlocal Damping and Nonlinear Boundary Damping

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Abstract. In this paper, we consider a nonlinear wave equation with nonlocal damping and nonlinear boundary damping. We prove a general energy decay property for solutions in terms of coefficient of the frictional boundary damping by using of the multiplier technique from the idea of Martinez [1]. Our result extends and improves the result in the literature such as the work by Lourêdo, Ferreira de Araújo and Mirandain [2] in which only exponential energy decay is considered. Furthermore, we get also the energy decay for the equation with nonlocal damping only but without nonlinear boundary damping.

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Key Words: Wave equation; general decay; nonlocal damping; boundary damping.

1 Introduction

In this paper, we consider the mixed problem for the wave equation with nonlocal damping and nonlinear boundary damping

$$u_{tt}-\Delta u+M\left(\int_\Omega |u_t|^2 \mathrm{d}x\right)u_t+N\left(\int_\Omega |u|^p \mathrm{d}x\right)|u|^{p-2}u=0, \qquad \text{in } \Omega\times(0,+\infty), \tag{1.1}$$

$$u=0, \qquad \text{on } \Gamma_0\times(0,+\infty), \tag{1.2}$$

$$\frac{\partial u}{\partial \nu}+\alpha(t)h(u_t)=0, \qquad \text{on } \Gamma_1\times(0,+\infty), \tag{1.3}$$

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \qquad \text{on } \Omega, \tag{1.4}$$

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where Ω is a bounded domain of $R^n(n\geq 1)$ with sufficiently smooth boundary $\partial\Omega=\Gamma$ such that $\Gamma=\Gamma_0\cup\Gamma_1$, $\bar{\Gamma}_0\cap\bar{\Gamma}_1=\emptyset$ and Γ_0,Γ_1 have positive measures, M,N,α and h are real valued functions satisfying the general conditions (A1)-(A5) (specified below). The above problem was given by Lourêdo, Ferreira de Araújo and Mirandain in [2] when $p=2$ and $\alpha=1$, where u describe the meson field amplitude and the term $M(\int_\Omega|u_t|^2\mathrm{d}x)u_t$ gives an internal dissipation, and it was motivated by the works of Schiff [3] and Jorgens [4] on a nonlinear theory of meson fields. Lourêdo, Ferreira de Araújo and Mirandain [2] study the following problem

$$\begin{aligned}
&u_{tt}-\mu(t)\Delta u+\alpha f\left(\int_\Omega|u|^2\mathrm{d}x\right)u+\beta g\left(\int_\Omega|u_t|^2\mathrm{d}x\right)u_t=0, && \text{in } \Omega\times(0,+\infty),\\
&u=0, && \text{on } \Gamma_0\times(0,+\infty),\\
&\frac{\partial u}{\partial\nu}+h(\cdot,u_t)=0, && \text{on } \Gamma_1\times(0,+\infty),\\
&u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), && \text{on } \Omega,
\end{aligned}$$

where the functions $\mu(t),f(s),g(s)$ satisfy the conditions $\mu(t)\geq\mu_0>0,f(s)\geq 0,g(s)\geq 0$ for $t\geq 0,s\geq 0$; $h(x,s)$ is a real function where $x\in\Gamma_1,s\in R$ and α,β are non-negative real constants. They show the global existence of solutions by Galerkin method under the following conditions on $h=h(x,u_t)$: $h(x,0)=0,|h(x,s)|\leq d_1|s|$ and $h(x,s)s\geq d_0s^2,\forall s,r\in R$. They also obtain the exponential decay of the energy by two methods: by using a Lyapunov functional and by Nakao's method. The nonlocal nonlinear term in the present work can be seen as a generalization of such kind of sources. In this paper we aim to investigate the exponential decay of the energy for problem (1.1)-(1.4).

In the more recent papers [5–7], Haraux et al give also an example to wave equation with nonlinear averaged damping terms of the form

$$u_{tt}(t,x)+\left(\int_\Omega u_t^2(t,x)\mathrm{d}x\right)^{\frac{\alpha}{2}}u_t(t,x)-\Delta u(t,x)=h(t,x)$$

in a bounded domain Ω with homogeneous Dirchlet boundary conditions. They investigate the asymptotic behavior of solutions for an abstract second-order equation with a linear "elastic" part and a nonlinear damping term depending on a power of the norm of the velocity.

When $N=$constant and $M=$constant, problems like (1.1)-(1.4) has been widely studied, we refer to [8–15] and reference therein, where under some assumptions imposed on the interior and boundary sources and damping terms and initial data, global existence, global nonexistence and decay properties for the solutions of problems (1.1)-(1.4) were proved.

We note that nonlocal sources have been recently approached in the theory of plates. Indeed, in Khanmamedov and Simsek [16] it is considered the following plate equation

$$u_{tt}+\Delta^2u+\alpha(x)u_t+\lambda u+f\left(\int_{R^n}|u|^p\mathrm{d}x\right)|u|^{p-2}u=h(x) \text{ in } R^n\times(0,+\infty). \tag{1.5}$$