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LONG TIME STABILITY OF A LINEARLY EXTRAPOLATED BLENDED BDF SCHEME FOR MULTIPHYSICS FLOWS

AYTEKIN ÇIBIK, FATMA G. EROGLU, AND SONGUL KAYA*

Abstract. This paper investigates the long time stability behavior of multiphysics flow problems, namely the Navier-Stokes equations, natural convection and double-diffusive convection equations with an extrapolated blended BDF time-stepping scheme. This scheme combines the two-step BDF and three-step BDF time stepping schemes. We prove unconditional long time stability theorems for each of these flow systems. Various numerical tests are given for large time step sizes in long time intervals in order to support theoretical results.

Key words. Blended BDF, long time stability, Navier-Stokes, natural convection, double-diffusive.

1. Introduction

Most of the engineering and applied science problem involves the combination of some different physical problems such as fluid flow, heat transfer, mass transfer, and electromagnetic effect. These kinds of problems are mostly referred as multiphysics problems. From a mathematical point of view, these problems yield systems of coupled single physics equations. In our case, many important applications require the accurate solution of multiphysics coupling with Navier-Stokes equations. Since the simulation of Navier-Stokes equations has its own difficulties, the coupling among involved equations yield more complex problems. One possibility of improving numerical simulations is to develop algorithms which are reliable and robust. In addition, designed numerical schemes should capture the long time dynamics of the system in a right way. Thus, it is of practical interest to have an algorithm which is stable over the required long time intervals.

In recent years, considerable amount of effort has been spent to understand the long time behavior of the numerical schemes for multiphysics problems. For such works, we refer to [3, 6, 19, 20, 22, 24]. In particular, for Navier-Stokes equations, the Crank-Nicholson in [11, 12, 19], the implicit Euler in [22], two-step Backward Differentiation (BDF2) in [2] and fractional step/projection methods in [18] are chosen as temporal discretization in order to show the long time stability. In this respect, an extrapolated two step BDF scheme for a velocity-vorticity form of Navier-Stokes equations has been investigated in [10]. The long time stability of partitioned methods for the fully evolutionary Stokes-Darcy problem in [13], for the time dependent MHD system in [14, 20] and for the double-diffusive convection in [21] were also established based on implicit-explicit and backward differentiation schemes.

This study concerns the behavior of the solutions of multiphysics problems for longer time simulations and time step restrictions on these solutions. In order to solve the systems numerically, a finite element in space discretization is employed along with a rather new time stepping approach so called an extrapolated

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*Corresponding Author.

blended Backward Differentiation (BLEBDF) temporal discretization. Such kind of a scheme combines BDF2 and a three-step BDF method in order to not only preserve A-stability and second order accuracy but also have a smaller constant in truncation error terms, [23]. Along with the mentioned time-stepping strategy, the three-step extrapolation is used in order to linearize the convective terms in the system of PDE's. Thus, the solution of only one linear system of equations is encountered at each time step which reduces the compilation time and memory cost in simulations.

In [17], Ravindran considers the stability and convergence of a double diffusive convection system with the blended BDF scheme in short time intervals. The current study attempts to extend the works above to study the notion of long time stability by combining the BLEBDF idea and its effects on several multiphysics flows such as Navier-Stokes, natural convection and double-diffusive convection. In this work, we will provide the unconditonal long time L^2 stability property of BLEBDF method for each of flow systems, when they are discretized spatially with finite element method. To the best of authors' knowledge, this is the first study that investigates the long time stability of the finite element solutions of multiphysics flows involving a blended BDF time-stepping approach along with a third order linear extrapolation idea.

The plan of the paper is as follows. In Section 2, we state the notations with some mathematical preliminaries. Section 3 is reserved for proving the unconditional long time stability of Navier-Stokes equations under the employment of extrapolated blended BDF temporal discretization along with numerical experiments. Similar results are established for natural convection and double-diffusive convection equations in Section 4 and Section 5, respectively. Finally, we state some conclusion remarks in the last section.

2. Notations and Preliminaries

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be open, connected domain bounded by Lipschitz boundary $\partial\Omega$. Throughout the paper standard notations for Sobolev spaces and their norms will be used, c.f. Adams [1]. The norm in $(H^k(\Omega))^d$ is denoted by $\|\cdot\|_k$ and the norm in Lebesgue spaces $(L^p(\Omega))^d$, $1 \le p < \infty$, $p \neq 2$ by $\|\cdot\|_{L^p}$ and $p=\infty$ by $\|\cdot\|_\infty$. The space $L^2(\Omega)$ is equipped with the norm and inner product $\|\cdot\|$ and $(\cdot,\cdot)$, respectively, and for these we drop the subscripts. Vector-valued functions will be identified by bold face. The norm in dual space H^{-1} of $H_0^1(\Omega)$ is denoted by $\|\cdot\|_{-1}$. The continuous velocity, pressure, temperature and concentration spaces are denoted by

$$\mathbf{X} \;:=\; (\mathbf{H}_0^1(\Omega))^d,\; Q := L_0^2(\Omega),\; W := H_0^1(\Omega),\; \Psi := H_0^1(\Omega),$$

and the divergence free space

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X} : (\nabla \cdot \mathbf{v}, q) = 0, \forall q \in Q\}.$$

We recall also the Poincaré-Friedrichs inequality as

$$\|\mathbf{v}\| \le C_P \|\nabla \mathbf{v}\|, \quad \forall \mathbf{v} \in \mathbf{X}.$$

For each multiphysics flow problems, we consider a regular, conforming family Π^h of triangulations of domain with maximum diameter h for spatial discretization. Assume $\mathbf{X}_h \subset \mathbf{X}, Q_h \subset Q, W_h \subset W$ and $\Psi_h \subset \Psi$ be finite element spaces such that the spaces $(\mathbf{X}_h, Q_h)$ satisfy the discrete inf-sup condition needed for stability of the discrete pressure, [5]. The discretely divergence free space for $(\mathbf{X}_h, Q_h)$ pair