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A NEW APPROXIMATION ALGORITHM FOR THE MATCHING DISTANCE IN MULTIDIMENSIONAL PERSISTENCE*

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Abstract

Topological Persistence has proven to be a promising framework for dealing with problems concerning shape analysis and comparison. In this contexts, it was originally introduced by taking into account 1-dimensional properties of shapes, modeled by real-valued functions. More recently, Topological Persistence has been generalized to consider multidimensional properties of shapes, coded by vector-valued functions. This extension has led to introduce suitable shape descriptors, named the multidimensional persistence Betti numbers functions, and a distance to compare them, the so-called multidimensional matching distance.

In this paper we propose a new computational framework to deal with the multidimensional matching distance. We start by proving some new theoretical results, and then we use them to formulate an algorithm for computing such a distance up to an arbitrary threshold error.

Mathematics subject classification: 65D18, 68U05, 55N99.
Key words: Multidimensional persistent topology, Matching distance, Shape comparison.

1. Introduction

In this paper, we present a computational framework for applying some tools coming from *multidimensional persistence* to shape analysis and comparison. Indeed, interpreting and comparing shapes are probably two of the most challenging issues in the fields of Computer Vision, Computer Graphics and Pattern Recognition. Nowadays, shape models convey a great amount of visual, semantic and digital information, and therefore finding suitable methods allowing for capturing, processing and representing such an information in a convenient way is definitely a desirable target [43, 44].

Persistence for shape analysis and comparison. In this context, methods deriving from Topological Persistence have recently gained a growing appeal. They focus on a topological exploration of the data under study, with respect to some geometrical properties considered relevant for capturing the most salient features [3, 7, 25, 30, 46]. The assumption here is that the most important piece of information enclosed in geometrical data is usually the one that is "persistent" with respect to the defining parameters. More formally, the key idea is to model a shape as a space X, together with a real-valued function $\varphi : X \to \mathbb{R}$, called *filtering function.*

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The function φ plays the role of a descriptor for a shape property we consider relevant for the comparison or the analysis problem at hand. By studying the sublevel sets induced on X by φ, we can perform a topological exploration of the shape under study, focusing on the occurrence of meaningful topological events (e.g. the birth, or the death, of connected components, tunnels and voids). Such an information can then be encoded in a parameterized version of the Betti numbers, known in the literature as *persistent Betti numbers* [26], a *rank invariant* [10], and, for the 0th homology, a *size function* [28,31,45]. The main point is that these shape descriptors can be represented in a very simple and compact way, by means of the so-called *persistence diagrams*. Moreover, they are stable with respect to a suitable distance, i.e. the *bottleneck distance* or *matching distance*. Thus, the tools offered by Topological Persistence nicely fit for dealing with shape analysis and comparison problems, also supported by a set of computational frameworks which have been developed for applications [38]. Actually, in the last twenty-five years methods based on the previous guidelines have been successfully used in quite a lot of applications concerning shape analysis and comparison, see e.g. [4,11,14,17,22–24,36,45].

Motivations and prior works. A common scenario in applications is when two or more properties concur to define the shape of an object. Moreover, sometimes it is desirable to study properties of a shape that are intrinsically multidimensional, such as the coordinates of a point in the real plane, or the representation of colors in the RGB model. These considerations have driven the attention to the so-called *multidimensional Topological Persistence* [7,30]. Here the term multidimensional, or equivalently n-dimensional, refers to the fact that the considered filtering functions take values in $\mathbb{R}^n$.

Multidimensional persistence was firstly investigated in [29] as regards homotopy groups; multidimensional persistence modules were then considered in [9,10], and subsequently studied in other papers including the recent [34,35,42]. Another approach to the multidimensional setting is based on the so-called *foliation method*, proposed in [2] and used to analyse the multidimensional extension of persistent Betti numbers, namely the *n-dimensional persistent Betti numbers*, hereafter n-dimensional PBNs. Focusing on the 0th homology, the authors proved that, when $n > 1$, a foliation in half-planes can be given, such that the restriction of the n-dimensional 0th PBNs to these half planes turns out to be 1-dimensional. This allowed the definition of a proven stable matching distance between n-dimensional PBNs, namely the *n-dimensional matching distance*. Such a result has been partially extended in [5], i.e. for any homology degree but restricted to the case of *max-tame* filtering functions, and then further refined in [13] for continuous filtering functions. More recently, the *interleaving distance* between multidimensional persistence modules has been formally introduced and discussed in [34].

From the point of view of applications, a major issue in the multidimensional setting is that, when filtering functions are vector-valued, it is not possible to provide a complete and discrete representation for multidimensional persistence analogous to that provided by persistence diagrams [9].

Until now, the arising computational difficulties have been faced according to different strategies [8, 12, 19, 47], but the work is still in progress: for example, the question of if and how the interleaving distance between n-dimensional persistence modules can be efficiently computed or approximated remains open, as remarked in [34].

Contribution of the paper. We propose a solution to the problem of obtaining good approximations for the n-dimensional matching distance between kth PBNs. The first ideas in this