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Decoupled, Energy Stable Numerical Scheme for the Cahn-Hilliard-Hele-Shaw System with Logarithmic Flory-Huggins Potential

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Abstract. In this paper, a decoupling numerical method for solving Cahn-Hilliard-Hele-Shaw system with logarithmic potential is proposed. Combing with a convex-splitting of the energy functional, the discretization of the Cahn-Hilliard equation in time is presented. The nonlinear term in Cahn-Hilliard equation is decoupled from the pressure gradient by using a fractional step method. Therefore, to update the pressure, we just need to solve a Possion equation at each time step by using an incremental pressure-correction technique for the pressure gradient in Darcy equation. For logarithmic potential, we use the regularization procedure, which make the domain for the regularized functional $F(\phi)$ is extended from $(-1,1)$ to $(-\infty,\infty)$. Further, the stability and the error estimate of the proposed method are proved. Finally, a series of numerical experiments are implemented to illustrate the theoretical analysis.

AMS subject classifications: 35Q30, 74S05

Key words: Logarithmic potential, Cahn-Hilliard-Hele-Shaw, decoupling.

1 Introduction

Let $\Omega\subset R^d$, $d=2,3$ be an open polygonal or polyhedral domain with a Lipschitz continuous boundary $\partial\Omega$. The Cahn-Hilliard-Hele-Shaw (CH-HS) system can be expressed as

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following:

$$\begin{cases} \partial_t\phi+\nabla\cdot(\phi\mathbf{u})=\epsilon\Delta\mu & \text{in } \Omega\times(0,T], & (1.1a)\\ \mu=\dfrac{1}{\epsilon}f(\phi)-\epsilon\Delta\phi & \text{in } \Omega, & (1.1b)\\ \mathbf{u}=-(\nabla p+\gamma\phi\nabla\mu) & \text{in } \Omega, & (1.1c)\\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega, & (1.1d)\\ \phi(t=0)=\phi_0 & \forall\,\mathbf{x}\in\Omega, & (1.1e)\\ \partial_n\phi=\partial_n\mu=0,\mathbf{u}\cdot\mathbf{n}=0, & \text{on } \partial\Omega\times(0,T], & (1.1f)\end{cases}$$

where ϕ is the concentration field, $\mathbf{u}$ is the advective velocity, and p is the pressure, $\mathbf{n}$ is the unit outer normal of the boundary $\partial\Omega$, γ, ϵ are positive constants.

Noting that the first two equations in the system (1.1) can be seen as the Cahn-Hilliard (CH) equation with convertive term, and Eqs. (1.1c) and (1.1d) are considered as the Darcy equation with the elastic forcing term. The CH-HS system can be regarded as Cahn-Hilliard-Darcy (CH-D) equation while given thought to permeability/hydraulic conductivity in CH-HS system, which is used to model multi-phase flow in porous media.

The logarithmic free energy density function is defined as [1]

$$F(\phi)=\frac{\theta}{2}((1+\phi)\ln(1+\phi)+(1-\phi)\ln(1-\phi))+\frac{1}{2}(1-\phi^2),\quad \phi\in(-1,1), \tag{1.2}$$

and satisfied $\theta<1$.

The CH-HS system which is used to describe the motion of a viscous fluid between two closely spaced parallel plates is a very important mathematical model. The CH-HS system can be derived from the Navier-Stokes in the Hele-Shaw cell [2, 3]. The CH-HS system is used in many different applications, such as the process of the phase separation, tumor growth, cell sorting and multi-phase flows in porous media. In two dimension, the uniqueness of weak solution and the instantaneous propagation of regularity have been addressed in [4].

For CH-HS system with double-well potential, Wise proposed an unconditionally stable multi-grid method combining with finite difference method in [5]; Guo, Xia and Xu given a semi-implicit energy stable numerical scheme by using local discontinuous Galerkin method in [6]; By virtue of decoupling technique, Han proposed and analyzed a unconditionally stable mixed finite element method for the CH-HS system with variable viscosity and mobility in [7]. What is more, a second energy stable numerical scheme for CH-HS system was studied in [8] and the detailed convergence analysis can be found in recent works [9–12].

The coupling of the CH equation and other models is a valuable project, such as in [13], Bonfoh considered the CH-Gurtin model with logarithmic potential and obtained some results on the existence of solutions. Kay, Styles, and Welford made research on