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A Finite Volume Scheme for Savage-Hutter Equations on Unstructured Grids

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Abstract. A Godunov-type finite volume scheme on unstructured grids is proposed to numerically solve the Savage-Hutter equations in curvilinear coordinate. We show the direct observation that the model isn't a Galilean invariant system. At the cell boundary, the modified Harten-Lax-van Leer (HLL) approximate Riemann solver is adopted to calculate the numerical flux. The modified HLL flux is not troubled by the lack of Galilean invariance of the model and it is helpful to handle discontinuities at free interface. Rigidly the system is not always a hyperbolic system due to the dependence of flux on the velocity gradient. Even though, our numerical results still show quite good agreements to reference solutions. The simulations for granular avalanche flows with shock waves indicate that the scheme is applicable.

AMS subject classifications: 65M10, 65M15, 65N30

Key words: Granular avalanche flow, Savage-Hutter equations, finite volume method, Galilean invariant.

1. Introduction

The Savage-Hutter model was first proposed to describe the motion of a finite mass of granular material flowing down a rough incline, in which the granular material was treated as an incompressible continuum [8, 13, 28]. It was then extended to 2D and with curvilinear coordinate [9, 10, 12] which may be applied to study natural disasters as landslides and debris flows [20]. For more details on Savage-Hutter model, we refer to [7, 13–15, 25, 26].

Numerical method for the Savage-Hutter model may be dated back to work in [17, 28] using finite difference methods which are not able to capture shock waves. Later on, regardless of the Savage-Hutter model or other avalanches of granular flow models,

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the Godunov-type schemes are commonly used in literature [1, 2, 4, 27, 34, 35, 37] considering that most of them have hyperbolic nature of the depth averaged equations. We note that rigidly, the Savage-Hutter model is not hyperbolic since its flux is depended on the gradient of velocity. Precisely, the dependence is on the sign of the gradient of velocity components, that the model does be hyperbolic in the region with monotonic velocity components. Some related interesting development on numerical methods can be found in such as [1, 19, 21, 22, 30]. In the paper, we will apply a second-order finite volume method to simulating Savage-Hutter equations owe to its attactive property of local conservation for physical quantites [5, 11, 23, 31, 34, 36, 37].

In practical scenario, landslides, rock avalanches and debris flows usually occur in mountainous areas with complex topography, that we have to use unstructured grids for numerical simulation. Up to now, there are many numerical works on orthogonal structured grids, such as [1, 34, 35, 37]. However, there is few work of numerical methods for Savage-Hutter equations on unstructured grids. This may due primarily to the lack of Galilean invariance of the model, which is a direct observation to the model as we point out in Section 2. In this paper, we are motivated to develop a high resolution and robust numerical model for Savage-Hutter equations under the unstructured grids. Whether the numerical method may provide a correct solution on the unstructured grids for a problem without Galilean invariance is a major question here. Meanwhile, the loss of hyperbolicity at the zeros of velocity gradients may give rise to some unpredictable behavior in numerical solutions, too.

Additionally, there are still some challenges in numerical solving strongly convective Savage-Hutter equations on unstructured grids. For example, shock formation is an essential mechanism in granular flows on an inclined surface merging into a horizontal run-out zone or encountering an obstacle when the velocity becomes subcritical from its supercritical state [35]. Therefore, numerical efficiency is an important issue to help resolve the steep gradients and moving fronts. Noticing that the Savage-Hutter equations and Shallow water equations have some similarities, we basically follow the method in [3] for shallow water equations. The modified Harten-Lax-van Leer (HLL) flux is adopted to calculate the numerical flux on the cell boundary. The formation of the modified HLL flux does not require the flux to be Galilean invariant, thus formally we have no difficulties in calculation of numerical flux. The techniques involved to improve efficiency include the MUSCL-Hancock scheme in time discretization, the ENO-type reconstruction and the h-adaptive method in spatial discretization.

We apply the numerical scheme to three examples. The first one is a granular dam break problem, and the second one is a granular avalanche flows down an incline plane and merges continuously into a horizontal plane. In the third example, a granular slides down an inclined plane merging into a horizontal run-out zone is simulated, whereby the flow is diverted by an obstacle which is located on the inclined plane. It is interesting that in these examples, the numerical solutions are agreed with the references solutions quite well. We are indicated that the scheme is applicable to the model, though the model is lack of the Galilean invariance and is not always hyperbolic. The reason why the loss of the hyperbolicity of the model are not destructive to the