Numer. Math. Theor. Meth. Appl. Vol. **13**, No. 3, pp. 595-619
doi: 10.4208/nmtma.OA-2019-0175 August 2020

A High-Order Kernel-Free Boundary Integral Method for Incompressible Flow Equations in Two Space Dimensions

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Received 8 November 2019; Accepted (in revised version) 9 December 2019

Abstract. This paper presents a fourth-order kernel-free boundary integral method for the time-dependent, incompressible Stokes and Navier-Stokes equations defined on irregular bounded domains. By the stream function-vorticity formulation, the incompressible flow equations are interpreted as vorticity evolution equations. Time discretization methods for the evolution equations lead to a modified Helmholtz equation for the vorticity, or alternatively, a modified biharmonic equation for the stream function with two clamped boundary conditions. The resulting fourth-order elliptic boundary value problem is solved by a fourth-order kernel-free boundary integral method, with which integrals in the reformulated boundary integral equation are evaluated by solving corresponding equivalent interface problems, regardless of the exact expression of the involved Green's function. To solve the unsteady Stokes equations, a four-stage composite backward differential formula of the same order accuracy is employed for time integration. For the Navier-Stokes equations, a three-stage third-order semi-implicit Runge-Kutta method is utilized to guarantee the global numerical solution has at least third-order convergence rate. Numerical results for the unsteady Stokes equations and the Navier-Stokes equations are presented to validate efficiency and accuracy of the proposed method.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Unsteady Stokes equations, Navier-Stokes equations, stream function-vorticity formulation, kernel-free boundary integral method, composite backward difference formula, semi-implicit Runge-Kutta method.

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1. Introduction

In the incompressible viscous flow simulations, the Navier-Stokes equations take the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{f}, \tag{1.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.1b}$$

where $\mathbf{u}$ is the fluid velocity, p is the pressure, ρ is the density, ν is the kinematic viscosity and $\mathbf{f}$ is an external body force per unit mass. The second equation in (1.1) holds under the incompressible assumption that the fluid density ρ is uniform. Once the convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is neglected for low Reynolds flow, while the dependency of $\mathbf{u}$ on time evolution $\partial \mathbf{u}/\partial t$ is retained, the Navier-Stokes equations (1.1) are linearized to the following unsteady Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \nabla p = \mathbf{f}, \tag{1.2a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2b}$$

which describes motion of viscous incompressible flow with slow velocity and large viscosity. Eqs. (1.1) and (1.2) are completed by initial and boundary conditions for the velocity field $\mathbf{u}$, whereas no conditions are prescribed for the pressure field p. This leads to a difficulty of updating the unknown pressure in time advancing.

There are two mainstream ways in overcoming this issue. One is to eliminate the pressure term from the equations by taking curl over the governing equation, then the divergence constraint can be simply substituted and naturally satisfied [1–3]. Another is to interpret the pressure as the counterpart in equilibrium with the time-dependent divergence-free velocity field. In this way, velocity and pressure gradient are often connected by certain approximate or artificial divergence constraint and updated in a time-splitting scheme [4–7]. The method used in the current work is based on the former type, the so-called stream function-vorticity formulation.

The formulation is a coupled system, consisting of a nonlinear vorticity transport equation and a stream-function Poisson equation. The problem is often supplied with a no-slip boundary condition (BC), which gives a homogeneous expression for both the stream function and its normal derivative. However, no BCs are prescribed for the vorticity. A simple attempt is to update the vorticity only at the interior grid points by some explicit methods such as the time-forward or the Dufort-Frankel scheme so that boundary values of the vorticity are not required [1, 8, 9]. Then the stream function at new time step can be obtained by solving the Poisson equation with the updated vorticity as the source term. This method has limited development in the early years. Intensive efforts were made with implicit schemes, which however pose difficulty in boundary vorticity approximation. There are two classes of BCs for the vorticity transport equation. Those represented by Thoms formula [10], Woods's formula [11] and