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A Continuous-Stage Modified Leap-Frog Scheme for High-Dimensional Semi-Linear Hamiltonian Wave Equations

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Abstract. Among the typical time integrations for PDEs, Leap-frog scheme is the well-known method which can easily be used. A most welcome feature of the Leap-frog scheme is that it has very simple scheme and is easy to be implemented. The main purpose of this paper is to propose and analyze an improved Leap-frog scheme, the so-called continuous-stage modified Leap-frog scheme for high-dimensional semi-linear Hamiltonian wave equations. To this end, under the assumption of periodic boundary conditions, we begin with the formulation of the nonlinear Hamiltonian equation as an abstract second-order ordinary differential equation (ODE) and its operator-variation-of-constants formula (the Duhamel Principle). Then the continuous-stage modified Leap-frog scheme is formulated. Accordingly, the convergence, energy preservation, symplecticity conservation and long-time behaviour of explicit schemes are rigorously analysed. Numerical results demonstrate the remarkable advantage and efficiency of the improved Leap-frog scheme compared with the existing mostly used numerical schemes in the literature.

AMS subject classifications: 65M12, 65M20, 65M70, 65P10
Key words: Leap-frog scheme, Hamiltonian wave equations, modified Leap-frog scheme, continuous-stage methods.

1. Introduction

It is well known that Hamiltonian wave equations occur frequently in many branches of scientific fields such as acoustics, solid state physics, fluid dynamics, plasma physics,

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electromagnetics, nonlinear optics and quantum field theory (see, e.g., [21]). Hamiltonian wave equations have been found in a variety of nonlinear partial differential equations (PDEs) such as the sine-Gordon (SG) equation, the Klein-Gordon (KG) equation and the Korteweg and de Vries equation. The efficient and accurate numerical solutions of wave equations are of fundamental importance and have received a great deal of attention in recent decades, such as finite elements methods (see, e.g., [10,12,18,25,53,63]), trigonometric methods (see, e.g., [17,40]), energy-preserving methods (see, e.g., [13,20,24,58,59]), waveform relaxations (see, e.g., [22]), symplectic methods (see, e.g., [57]), spectrally accurate space-time solutions (see, e.g., [7]) and other methods (see, e.g., [3,4,19,35,60]).

Among these methods, Leap-frog scheme is mostly used in the time discretizations of PDEs and it has been researched in many publications (see, e.g., [32–34,47,49]). It is well known that Leap-frog discretizations of PDEs may have unbounded solutions for any choice of mesh-sizes even for choices satisfying conditions for linear stability [47]. This algorithm does, however, not behave well for wave equations with some space discretizations with a high degree of precision. Here we propose and analyze an improvement, which we name *modified Leap-frog scheme*. This improved scheme allows us to preserve different structures of the considered system. It is noted that the energy conservation and symplecticity are two key features of Hamiltonian systems and numerical algorithms should respect these properties, in the sense of structure-preserving integrators. In this paper, after showing the convergence of the schemes, different modified Leap-frog methods will be formulated to preserve different geometric or physic properties such as energy preservation, symplecticity and long time numerical energy conservation of explicit methods.

With this premise, in this paper, we propose and analyse a modified Leap-frog scheme for the high-dimensional semi-linear Hamiltonian wave equation of the form

$$\begin{cases} u_{tt} - a^2\Delta u = f(u), & 0 < t \le T, \quad x \in \Omega, \\ u(x,0) = \varphi_1(x), \quad u_t(x,0) = \varphi_2(x), & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the wave displacement at time t and position $x \in \Omega$ with $\Omega := (0, X_1) \times \cdots \times (0, X_d) \subset \mathbb{R}^d$, a is a real parameter, $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$, and $f(u)$ is the negative derivative of a smooth potential energy $G(u) \ge 0$ which can be expressed as

$$f(u) = -G'(u).$$

Eq. (1.1) is supplemented with the following periodic boundary conditions

$$u(x,t)|_{\partial\mathbf{\Omega}\cap\{x_j=0\}} = u(x,t)|_{\partial\mathbf{\Omega}\cap\{x_j=X_j\}}, \qquad j = 1, \cdots, d. \tag{1.2}$$

It is noted that the problem (1.1) is a Hamiltonian PDE of the form

$$\begin{cases} u_t = v, \\ v_t = a^2\Delta u + f(u), \end{cases} \tag{1.3}$$