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# A Fully Discrete Explicit Multistep Scheme for Solving Coupled Forward Backward Stochastic Differential Equations

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**Abstract.** In this work, we are concerned with the explicit multistep scheme for solving the coupled forward backward stochastic differential equations (FBSDEs). Based on the Lagrange interpolation and first-order derivative approximations, we will propose a fully discrete explicit high-order multistep scheme for solving coupled FBSDEs. Its high accuracy, efficiency and stability are verified by the numerical experiments.

**AMS subject classifications**: 65C20, 65C30, 60H35

**Key words**: Lagrange interpolation, derivative approximation, coupled forward backward stochastic differential equations, explicit multistep scheme.

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## 1 Introduction

Let $(\Omega,\mathcal{F},\mathbb{F},P)$ be a filtered complete probability space with the natural filtration $\mathbb{F}=\{\mathcal{F}_t\}_{0\le t\le T}$, generated by the $m$-dimensional standard Brownian motion $W=(W_t)_{0\le t\le T}$. We are concerned with the numerical solutions of fully coupled FBSDEs on $(\Omega,\mathcal{F},\mathbb{F},P)$ in the form

$$\begin{cases} X_t = X_0 + \int_0^t b(s,X_s,Y_s,Z_s)ds + \int_0^t \sigma(s,X_s,Y_s,Z_s)dW_s, \\ Y_t = \varphi(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)\mathrm{ds} - \int_{\mathrm{t}}^{\mathrm{T}} \mathrm{Z_s dW_s}, \end{cases} \tag{1.1}$$

for $t\in[0,T]$, where $T>0$ is the deterministic terminal time, $X_0\in\mathcal{F}_0$ is the initial condition of SDE, $b:[0,T]\times\mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times m}\to\mathbb{R}^d$ and $\sigma:[0,T]\times\mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times m}\to\mathbb{R}^{d\times m}$ are the drift and diffusion coefficients, respectively, $\varphi:\mathbb{R}^d\to\mathbb{R}^p$ is the terminal function of $X_T$ for BSDE, and

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$f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^p\times\mathbb{R}^{p\times m}\rightarrow\mathbb{R}^p$ is the generator of the BSDE. The two stochastic integrals in (1.1) with respect to $W_s$ are of the Itô type. The triple $(X_t,Y_t,Z_t)$ is referred to as an $L^2$-adapted solution of the FBSDEs (1.1) if it is $\mathcal{F}_t$-adapted, square integrable and satisfies (1.1). If $b$ and $\sigma$ are independent of $Y_t$ and $Z_t$, we call Eqs. (1.1) decoupled FBSDEs.

The existence and uniqueness of the adapted solution of nonlinear BSDEs was first obtained by Pardoux and Peng in their original work [29] under some standard conditions. After that, Peng and Wu proved the existence and uniqueness of the solution of fully coupled FBSDEs with an arbitrarily large time duration in [28]. By now, applications of FBSDEs have been found in many kinds of fields such as the research on PDEs [12,20], stochastic optimal control [16, 26], mathematical finance [9, 11, 21, 22], mean-field BSDEs [31] and stochastic differential games [3, 5, 18, 32], etc. Therefore it is interesting and important to study the numerical solutions of FBSDEs.

So far, there have been a considerable number of numerical methods for solving BSDEs [6–9,17,37,40,41], decoupled FBSDEs [10,13,15,22,23,33,35,39,42] and second order forward backward stochastic differential equations (2FBSDEs) [14,30,36,43] in literatures. However, because of the complicated structure of the fully coupled FBSDEs, the numerical schemes for solving fully coupled FBSDEs [38] are very few up to now. By using the local property of the generator of diffusion processes, the authors in [38] constructed an implicit high order multistep scheme for solving the fully coupled FBSDEs (1.1) with the Euler method used to solve SDE in (1.1). The use of the Euler method for the forward SDE dramatically simplify the entire computational scheme proposed in [38], but it is still very time consuming for solving the fully coupled FBSDEs due to an iteration procedure of this implicit multistep scheme therein.

In this paper, we aim to develop a more efficient explicit multistep numerical scheme for solving fully coupled FBSDEs. To this end, we introduce the Lagrange interpolation method into the implicit scheme proposed in [38], and then propose a fully explicit multistep scheme for solving the coupled FBSDEs. In our proposed explicit scheme, we first use the Lagrange interpolation polynomial to predict the values of the unknown terms contained in the functions $b$, $\sigma$ and $f$. Then we avoid solving the implicit equations which include the iteration methods therein. Thus, our new proposed explicit scheme can reduce the computational time remarkably. To attest the stability, effectiveness and high accuracy of our scheme, we also carry out some numerical experiments. Our numerical results show that the explicit scheme is stable, high accurate, and high efficient for solving fully coupled FBSDEs.

The rest of the paper is organized as follows. In Section 2, some preliminaries such as the generator of diffusion process, Feynman-Kac formula and function approximation are introduced. In Section 3, we give two discrete reference equations, and based on which, we propose the explicit time semidiscrete scheme for solving coupled FBSDEs. Then we draw the time-space fully discrete scheme for coupled FBSDEs in Section 4. In Section 5, numerical experiments are carried out to demonstrate the effectiveness and the accuracy of the proposed schemes. Finally some conclusions are given in Section 6.