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# Global Solution and Exponential Stability for a Laminated Beam with Fourier Thermal Law

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**Abstract.** This paper focuses on the long-time dynamics of a thermoelastic laminated beam modeled from the well-established Timoshenko theory. From mathematical point of view, the study system consists of three hyperbolic motion equations coupled with the parabolic equation governed by Fouriers law of heat conduction and, in consequence, does not belong to one of the classical categories of PDE. We have proved the well-posedness and exponential stability of the system. The well-posedness is given by Hille-Yosida theorem. For the exponential decay we applied the energy method by introducing a Lyapunov functional.

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**Key Words**: Global solution; laminated beam; Timoshenko; thermoelasticity; energy method.

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## 1 Introduction

The one dimensional thermoelastic system is given by

$$\rho u_{tt} - a u_{xx} + \alpha \theta_x = 0, \tag{1.1}$$

$$c\theta_t - a\theta_{xx} + \alpha u_{xt} = 0. \tag{1.2}$$

In this model, $\rho$ denotes the mass density, $a$ the elasticity coefficient, a the stress-temperature and $c$ the heat conductivity. The functions $u$ and $\theta$ are the displacement of

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the solid elastic material and the temperature difference. For existence and the asymptotic stability of the solutions we cite the pioneer work of Dafermos [1] where it is proven that the temperature gradient and the specific entropy always converges to zero.

As a rule, the displacement also decays to zero as time goes to infinity. Several efforts have shown asymptotic stability, specifically [2, 3] and reference therein. In these studies, the authors proved that the total thermoelastic energy decays to zero exponentially as time goes to infinity for material subject to Dirichlets, Neumanns and also mixed boundary conditions. The beam deflection when subjected to transverse displacement $u$ and rotation angle $\psi$ is mathematically described by the system developed by Timoshenko [4], which is given by two coupled differential equations

$$\rho u_{tt}+G(\psi-u_x)_x=0, \tag{1.3}$$

$$I_\rho \psi_{tt}-G(\psi-u_x)-D\psi_{xx}=0. \tag{1.4}$$

The coefficients $\rho$, $I_\rho$ and $G$ are the mass per unit length, the polar moment of inertia of a cross section and the shear modulus, respectively. $D=EI$ where $E$ is Youngs modulus of elasticity and $I$ is the moment of inertia of a cross section.

The model for two identical Timoshenko beams, taking into account that an adhesive of the small thickness is bonding the two layers producing the structural damping due to the interfacial slip, was proposed by Hansen and Spies [5, 6] and is given by

$$\rho u_{tt}+G(\psi-u_x)_x=0, \qquad x\in(0,1), \quad t\geq 0, \tag{1.5}$$

$$I_\rho(3S-\psi)_{tt}-G(\psi-u_x)-D(3S-\psi)_{xx}=0, \qquad x\in(0,1), \quad t\geq 0, \tag{1.6}$$

$$3I_\rho S_{tt}+3G(\psi-u_x)+4\delta S+4\gamma S_t-3DS_{xx}=0, \qquad x\in(0,1), \quad t\geq 0, \tag{1.7}$$

where $u(x,t)$ represents the transverse displacement, $\psi(x,t)$ is the rotation angle displacement and $S(x,t)$ is proportional to the amount of slip along the interface. The system (1.5)-(1.7) describes the dynamics of transverse displacement, rotation angle and interfacial slip, respectively. The coefficients $\delta$ and $\gamma$ are the adhesive stiffness and adhesive damping of the beams.

Regarding the stabilization of the system (1.5)-(1.7), we mention [7–10] and references therein. In [9], it is proven that the structural damping $4\gamma S_t$ created by the interfacial slip alone is not enough to stabilize the system (1.5)-(1.7) exponentially to its equilibrium state. Reference [10] showed that when the frictional damping is present in all components,

$$\rho u_{tt}+G(\psi-u_x)_x+\alpha u_t=0, \tag{1.8}$$

$$I_\rho(3S_{tt}-\psi_{tt})-G(\psi-u_x)-D(3S_{xx}-\psi_{xx})+\beta(3S-\psi)_t=0, \tag{1.9}$$

$$3I_\rho S_{tt}+3G(\psi-u_x)+4\delta S-3DS_{xx}+4\gamma S_t=0, \tag{1.10}$$