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Multiple Axially Asymmetric Solutions to a Mean Field Equation on $\mathbb{S}^2$

Zhuoran Du[1], Changfeng Gui[2,*], Jiaming Jin[1] and Yuan Li[1]

[1] *School of Mathematics, Hunan University, Changsha, Hunan 410082, China*
[2] *Department of Mathematics, University of Texas at San Antonio, TX78249, USA*

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Abstract. We study the following mean field equation

$$\Delta_g u + \rho \left(\frac{e^u}{\int_{\mathbb{S}^2} e^u d\mu} - \frac{1}{4\pi} \right) = 0 \quad \text{in } \mathbb{S}^2,$$

where ρ is a real parameter. We obtain the existence of multiple axially asymmetric solutions bifurcating from $u = 0$ at the values $\rho = 4n(n+1)\pi$ for any odd integer $n \geq 3$.

Key Words: Mean field equation, axially asymmetric solutions, bifurcation.

AMS Subject Classifications: 35B32, 35J61, 58J55

1 Introduction

In this paper, we consider the mean field equation on the unit sphere

$$\Delta_g u + \rho \left(\frac{e^u}{\int_{\mathbb{S}^2} e^u d\mu} - \frac{1}{4\pi} \right) = 0 \quad \text{in } \mathbb{S}^2, \tag{1.1}$$

where ρ is a real parameter, Δ_g stands for Laplace-Beltrami operator on $\mathbb{S}^2$ associated to the metric g inherited from the ambient Euclidean metric and $d\mu$ is the volume form with respect to g. Since the above equation is invariant by adding a constant to a solution, we introduce

$$\mathcal{H} = \left\{ u \in H^2(\mathbb{S}^2) \Big| \int_{\mathbb{S}^2} u d\mu = 0 \right\},$$

*Corresponding author. *Email addresses:* `duzr@hnu.edu.cn` (Z. Du), `changfeng.gui@utsa.edu` (C. Gui), `jiamingjin123@163.com` (J. Jin), `liy93@hnu.edu.cn` (Y. Li)

where $H^2(\mathbb{S}^2)$ is the classical sobolev space. Note that $\mathcal{H}$, equipped with the $H^2(\mathbb{S}^2)$ norm, is a Hilbert space. We write $\mathbb{S}^2$ in the following coordinate system

$$\mathbb{S}^2=\left\{\left(\sqrt{1-z^2}\cos\theta,\ \sqrt{1-z^2}\sin\theta,\ z\right)\Big|\ z\in[-1,1],\ \theta\in[0,2\pi]\right\}.$$

Note that $\int_{\mathbb{S}^2} d\mu=4\pi$. Clearly equation (1.1) admits solution $u\equiv 0$, so in what follows mentioned existence of solutions of (1.1) means existence of non-trivial solution. The corresponding energy functional of (1.1) is

$$J_\rho(u)=\frac{1}{2}\int_{\mathbb{S}^2}|\nabla_g u|^2 d\mu-\rho\log\left(\int_{\mathbb{S}^2}e^u d\mu\right). \tag{1.2}$$

The study of existence of solution for mean field equations possess a long history and huge literature. This kind equations arise from Onsager's vortex model for turbulent Euler flows, see [20]. They also arise from the Chern-Simons-Higgs model vortex when some parameter tends to zero, we refer the reader to [3,25–27].

For $\rho<8\pi$, by the Moser-Truding inequality, J_ρ is bounded below and coercive, so the proof of the existence of a minimizer of J_ρ is standard. For $\rho=8\pi$, the existence of a minimizer of (1.2) had been proved in [21] and [7]. For $\rho>8\pi$, J_ρ is not bounded below. Topological degree theory plays an important role in the solvability of (1.1). Starting with the work of Li in [16], one knows that the solutions of (1.1) are uniformly bounded on any compact subset of $\cup_{m=0}^{\infty}(8m\pi,8(m+1)\pi)$, and the Leray-Schauder topological degree $d_\rho=1$ for $\rho<8\pi$. Due to the result of Li and the homotopy invariance of the degree, it is readily checked that d_ρ is constant in each interval $\rho\in(8m\pi,8(m+1)\pi)$. Further Lin in [17] proved that $d_\rho=-1$ for $8\pi<\rho<16\pi$, and $d_\rho=0$ for $16\pi<\rho<24\pi$. Subsequently Chen and Lin in [4] obtained apriori bound for a sequence ρ_n with $\rho=\rho_n$. Using this apriori bound, they were able to calculate the degree in [5] $d_\rho=0$ for $\rho\in(8m\pi,8(m+1)\pi),m\in\mathbb{N}$ with $m\geq 2$. By a more precise topological argument, the author of [8] proved that in this case (1.1) admits a solution for any $\rho\in\mathbb{R}\backslash 8\pi\mathbb{N}$. Dolbeault, Esteban and Tarantello in [9] proved that for all $k\geq 2$ and $\rho>4k(k+1)\pi$ (so $\rho>24\pi$), (1.1) admits at least $2(k-2)+1$ distinct axially symmetric solutions by using bifurcation method. Indeed, they proved that for any $k\geq 2$ there are two continuous unbounded half-branches of solutions of (1.1) bifurcating from the trivial solution at points $\rho=4k(k+1)\pi$. Note that blow-up solutions could appear only when $\rho\to 8m\pi$, $m\in\mathbb{N}$. Lin in [17] establish the existence of the blow-up solutions to (1.1) when ρ approaches 16π from above. Recently Gui and Hu [13] proved the existence of a family of blow-up solutions for ρ approaches 32π by using Lyapunov reduction method. As far as uniqueness is concerned, Lin in [18] showed that the solution to (1.1) is unique for $0<\rho<8\pi$, namely (1.1) only admits the trivial solution. Lin in [19] showed that the axially symmetric solution to (1.1) is unique for $8\pi<\rho\leq 16\pi$, namely $u\equiv 0$ is the only axially symmetric solution of (1.1). By developing a "sphere covering inequality", Gui and Moradifam in [14] extend the uniqueness result to a broader parameter range $\rho\in(0,8\pi)\cup(8\pi,16\pi)$ for any solutions of (1.1). By applying the "sphere covering