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Solving Fourth-Order PDEs using the LMAPS

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Abstract. To overcome the difficulty for solving fourth order partial differential equations (PDEs) using localized methods, we introduce and extend a recent method to decompose the particular solution of such equation into particular solutions of two second-order differential equations using radial basis functions (RBFs). In this way, the localized method of approximate particular solutions (LMAPS) can be used to directly solve a fourth-order PDE without splitting it into two second-order problems. The closed-form particular solutions for polyharmonic splines RBFs augmented with polynomial basis functions for Helmholtz-type equations are the cores of the solution process. Several novel techniques are proposed to further improve the accuracy and efficiency. Four numerical examples are presented to show the effectiveness of our approach.

AMS subject classifications: 65Y04, 35K05

Key words: Localized method of approximate particular solutions, Helmholtz-type operator, fourth-order partial differential equation, polynomial basis functions, polyharmonic splines of RBFs.

1 Introduction

During the past few decades, the traditional mesh-based methods such as the finite element method [1, 2, 19], the finite difference method [23, 24] and the finite volume

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method [17, 18] are the main stream numerical methods for solving challenging problems in science and engineering. In recent years, the development of meshless methods have grown substantially due to their simplicity for solving complicate domains, specially in high dimensions. In this paper, we will focus on solving fourth-order partial differential equations (PDEs) which play an important role in modeling a variety of physical phenomenon and have been widely studied in the area of science and engineering such as computer graphics [21], fluid in lungs [15] and plate bending problems [12], etc. In the past, many numerical methods have been developed for solving fourth-order PDEs [7–10]. One of the challenges for solving this type of equations using localized methods is the difficulty of approximating the fourth derivative in the governing equation using only a small number of neighboring points in the local influence domain. Traditionally, the given fourth-order PDE is often split into two second-order problems [3, 13, 16, 28] to avoid dealing with the approximation of higher order derivatives. On the other hand, a negative side of such approach is that the solution process is only limited to a small class of fourth-order PDEs, such as the biharmonic equation, with Dirichlet and Laplace boundary conditions.

In this paper, instead of splitting a fourth-order PDE into two decoupled second-order problems, we adopt a recently introduced method [4] in decomposing the particular solution of a given fourth-order PDE into the linear combination of particular solutions of two second-order problems. Furthermore, we extend the method in [16, 28] to the localized method of approximate particular solutions (LMAPS) using polyharmonic splines radial basis function (RBF) augmented with polynomial basis functions. We also propose a number of novelties to further improve the accuracy and stability. As a result, we are able to overcome the difficulty in [16, 28] and directly solve an extended class of fourth-order PDEs with general types of boundary conditions.

The LMAPS is known as an effective meshless collection method [25]. The successful implementation of the LMAPS depends on the availability of the closed-form particular solution of the corresponding PDE. However, similar to the derivation of the fundamental solution in the boundary element method, the derivation of the closed-form particular solution is not always possible and only a limited number of particular solutions for a general type of differential operators are available using radial basis functions [20, 22]. Recently, a general algorithm for the generation of closed-form particular solution using polynomial basis functions has been developed [6].

In this paper, we apply the known closed-form particular solutions of Helmholtz-type equations for both polyharmonic splines radial basis functions [20, 22] and polynomial basis functions [14] to form particular solutions of fourth-order PDEs without going through the tedious derivation. Based on these formulation, eventually we are able to extend these results to solve other fourth-order PDEs.

The organization of this paper is as follows. We briefly review the LMAPS in Section 2. In Section 3, we present how to reformulate particular solutions of the fourth-order PDEs using the closed-form particular solutions of Helmholtz-type equations. In Section 4, four numerical examples are presented to demonstrate the effectiveness of our proposed