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Blowup and Asymptotic Behavior of a Free Boundary Problem with a Nonlinear Memory

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Abstract. In this paper, we investigate a reaction-diffusion equation $u_t - du_{xx} = au + \int_0^t u^p(x,\tau)\mathrm{d}\tau + k(x)$ with double free boundaries. We study blowup phenomena in finite time and asymptotic behavior of time-global solutions. Our results show if $\int_{-h_0}^{h_0} k(x)\psi_1 \mathrm{d}x$ is large enough, then the blowup occurs. Meanwhile we also prove when $T^* < +\infty$, the solution must blow up in finite time. On the other hand, we prove that the solution decays at an exponential rate and the two free boundaries converge to a finite limit provided the initial datum is small sufficiently.

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Key Words: Nonlinear memory; free boundary; blowup; asymptotic behavior.

1 Introduction

In this paper, we consider the following one-dimension free boundary problem with a nonlinear memory

$$\begin{cases} u_t - du_{xx} = au + \int_0^t u^p(x,\tau)\mathrm{d}\tau + k(x), & g(t) < x < h(t),\ t > 0, \\ u(g(t),t) = 0,\ \ g'(t) = -\mu u_x(g(t),t), & t > 0, \\ u(h(t),t) = 0,\ \ h'(t) = -\mu u_x(h(t),t), & t > 0, \\ g(0) = -h_0,\ h(0) = h_0,\ u(x,0) = u_0(x), & -h_0 \le x \le h_0. \end{cases} \tag{1.1}$$

From a physical point of view, the differential equation in (1.1) with $a=0$, $p=1$, $k(x) \equiv 0$ appears in the theory of the nuclear reactor dynamics (see [1]). $x = g(t)$ and $x = h(t)$

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are moving boundaries to be determined together with the solution. In this paper, we assume $a\in\mathbb{R}$, $h_0>0$, $p>1$, d, $\mu>0$ and $k(x)>0$ is a smooth function. The initial function u_0 is chosen from $\mathfrak{X}(h_0)$ for some $h_0\in(0,\infty)$, where

$$\mathfrak{X}(h_0):=\Big\{u_0\in C^2([-h_0,h_0]:u_0>0 \text{ in } (-h_0,h_0) \text{ with } u_0(-h_0)=u_0(h_0)=0\Big\}. \tag{1.2}$$

For fixed domains, Bellout in [2] considered a similar equation

$$u_t-\Delta u=\int_0^t(u+\lambda)^p\mathrm{d}s+g(x) \text{ in } \Omega_T,$$

where $g(x)\geq 0$ is a smooth function and $\lambda>0$.

Li and Xie in [3] studied the following problem

$$\begin{cases} u_t-\Delta u=u^p\displaystyle\int_0^t u^q(x,\tau)\mathrm{d}\tau, & (x,t)\in\Omega\times(0,T),\\ u(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

they obtained the blowup criteria and the blowup rate. For other related works, we refer to [4–6] and the references therein.

The equation governing the free boundary, $h'(t)=-\mu u_x(h(t),t)$, is a special case of the well-known Stefan condition, which was given by Josef Stefan in his papers appeared in 1889. The original Stefan problem treats the formation of ice in the polar seas. Until now , the Stefan condition has been used in the modeling of a number of applied problems. For example, it was used to describe the melting of ice in contact with water [7], in the modeling of oxygen in the muscle [8], and in wound healing [9] and tumor growth [10–12]. There is a vast literature on the Stefan problem, and some important recent theoretical advances can be found in [8, 13]. As we know, the free boundary condition has been used in many areas, we can refer to several earlier papers, for example, [14–21] and the references therein.

The nonlocal parabolic equations including space-integral nonlocal source terms seem to be much more investigated than the corresponding nonlocal equations with a time-integral terms. Recently, in [22], the authors considered the following free boundary problem with space-integral nonlocal source:

$$\begin{cases} u_t-du_{xx}=a\displaystyle\int_{g(t)}^{h(t)}u^p\mathrm{d}x, & t>0,\ g(t)<x<h(t),\\ u(t,g(t))=0,\ g'(t)=-\mu u_x(t,g(t)), & t>0,\\ u(t,h(t))=0,\ h'(t)=-\mu u_x(t,h(t)), & t>0,\\ g(0)=-h_0,\ h(0)=h_0,\ u(0,x)=u_0(x), & -h_0\leq x\leq h_0. \end{cases} \tag{1.3}$$