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Higher Order Dissipative-Dispersive System and Application

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Abstract. In this paper, a generalized nonlinear dissipative and dispersive equation with time and space-dependent coefficients is considered. We show that the control of the higher order term is possible by using an adequate weight function to define the energy. The existence and uniqueness of solutions are obtained via a Picard iterative method. As an application to this general Theorem, we prove the well-posedness of the Camassa-Holm type equation.

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1 Introduction

1.1 Presentation of the problem

In this paper, we study the Cauchy problem for the general nonlinear higher order dissipative-dispersive equation:

$$\begin{cases} (1-m\partial_x^2)u_t+a_1(t,x,u)u_x+a_2(t,x,u,u_x)u_{xx}+a_3(t,x,u)u_{xxx} \\ \qquad +a_4(t,x)u_{xxxx}+a_5(t,x)u_{xxxxx}=f, \quad \text{for } (t,x)\in(0,T]\times\mathbb{R}, \\ u_{|_{t=0}}=u^0, \end{cases} \tag{1.1}$$

where $u=u(t,x)$, from $[0,T]\times\mathbb{R}$ into $\mathbb{R}$, is the unknown function of the problem, $m>0$, a_i, $1\leq i\leq 5$ and f are real-valued smooth given functions where their exact regularities will be

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precised later. This equation covers several important unidirectional models for the water wave problem at different regimes which take into account the variations of the bottom and the surface tension. We have in view in particular the example of the Camassa-Holm equation was first derived by Camassa and Holm in [1] (see also [2–4]), which is more nonlinear then the KdV and BBM equations (see for instance [5–11]). The presence of the fifth order derivative term is very important, so that the equation describes both nonlinear and dispersive effects as does the Camassa-Holm equation in the case of special tension surface values (see [12]).

Looking for solutions of (1.1) plays an important and significant role in the study of unidirectional limits for water wave problems with variable depth and topographies. To our knowledge the problem (1.1) has not been analyzed previously. In the present paper, we prove the local well-posedness of the initial value problem (1.1) by a standard Picard iterative scheme and the use of adequate energy estimates under a condition of nondegeneracy of the higher dispersive coefficient a_5. Therefore we apply this general theorem, to prove the well-posedness of the higher order Camassa-Holm-type equation.

1.2 Notations and main result

In the following, C_0 denotes any nonnegative constant whose exact expression is of no importance. The notation $a \lesssim b$ means that $a \leq C_0\, b$.

We denote by $C(\lambda_1,\lambda_2,\ldots)$ a nonnegative constant depending on the parameters λ_1, λ_2,... and whose dependence on the λ_j is always assumed to be nondecreasing.

For any $s \in \mathbb{R}$, we denote $[s]$ the integer part of s.

Let p be any constant with $1 \leq p < \infty$ and denote $L^p = L^p(\mathbb{R})$ the space of all Lebesgue-measurable functions f with the standard norm

$$|f|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \mathrm{d}x \right)^{1/p} < \infty.$$

The real inner product of any two functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R})$ is denoted by

$$(f_1,f_2) = \int_{\mathbb{R}} f_1(x) f_2(x) \mathrm{d}x.$$

The space $L^\infty = L^\infty(\mathbb{R})$ consists of all essentially bounded and Lebesgue-measurable functions f with the norm

$$|f|_{L^\infty} = \sup |f(x)| < \infty.$$

We denote by $W^{1,\infty}(\mathbb{R}) = \{f, \text{ s.t. } f, \partial_x f \in L^\infty(\mathbb{R})\}$ endowed with its canonical norm.

For any real constant $s \geq 0$, $H^s = H^s(\mathbb{R})$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_{L^2} < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1-\partial_x^2)^{1/2}$.

For any two functions $u = u(t,x)$ and $v(t,x)$ defined on $[0,T) \times \mathbb{R}$ with $T > 0$, we denote the inner product, the L^p-norm and especially the L^2-norm, as well as the Sobolev norm,