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Semi-Linear Fractional σ-Evolution Equations with Nonlinear Memory

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Abstract. In this paper we study the local or global (in time) existence of small data solutions to semi-linear fractional $\sigma-$evolution equations with nonlinear memory. Our main goals is to explain on the one hand the influence of the memory term and on the other hand the influence of higher regularity of the data on qualitative properties of solutions.

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1 Introduction

Fractional integrals and Fractional derivatives have applications in many fields including engineering, science, finance, applied mathematics, bio-engineering, radiative transfer, neutron transport, and the kinetic theory of gases, see, e.g. [1–3] to illustrate some applications. We refer also to the references [4, 5] for an introduction on the theory of fractional derivatives.

This note is devoted to the Cauchy problem for the semi-linear fractional σ-evolution equations with nonlinear memory. We are interested to the existence of solutions to the

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following Cauchy problem for the semi-linear fractional σ-evolution equations with non-linear memory

$$\begin{aligned} &\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = \int_0^t (t-s)^{-\mu} |u(s,\cdot)|^p \mathrm{d}s, \\ &u(x,0) = u_0(x), \quad u_t(0,x) = 0, \end{aligned} \tag{1.1}$$

where $\alpha \in (0,1), \sigma \geq 1, \mu \in (0,1), p > 1, (t,x) \in [0,\infty) \times \mathbb{R}^n$, $\partial_t^{1+\alpha} u = D_t^\alpha(u_t)$ with

$$D_t^\alpha(f) = \partial_t(I_t^{1-\alpha} f) \quad \text{and} \quad I_t^\beta f = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) \mathrm{d}s \text{ for } \beta > 0.$$

$D_t^\alpha(f)$ and $I_t^\beta f$ denotes the fractional Riemann-Liouville derivative and the fractional Riemann-Liouville integral respectively of f in $[0,t]$ and Γ is the Euler Gamma function. Our main goal is to understand on the one hand the improving influence of the nonlinear memory and on the other hand the influence of higher regularity of the data u_0 on the solvability behavior.

Remark 1.1. This problem has been studied in the case $\alpha = 1$ by several authors. We refere to reference [6] for the case $\sigma = 2$ and to references [7, 8] for the case $\sigma = 1$ with damped term. Also we refere to reference [9] for the case $\sigma = 1$ with structural damping.

In [10], Kainane and Reissig studied the following Cauchy problem for semi-linear fractional σ-evolution equations with power non-linearity

$$\begin{aligned} &\partial_t^{1+\alpha} u + (-\Delta)^\sigma u = |u|^p, \\ &u(x,0) = u_0(x), \quad u_t(0,x) = 0, \end{aligned} \tag{1.2}$$

where $\alpha \in (0,1)$, $\sigma \geq 1$. The authors proved the following results.

Proposition 1.1. *Let us assume $0 < \alpha < 1$, $\sigma \geq 1$ and $r \geq 1$. We assume that $n \geq \frac{2\sigma r}{1+\alpha}$. Moreover, the exponent p satisfies the condition*

$$\begin{aligned} &p > p_{\alpha,\sigma,r}(n) := \max\Big\{ p_{\alpha,\sigma}^r(n); \frac{1}{1-\alpha} \Big\}, \\ &\text{where } p_{\alpha,\sigma}^r(n) := 1 + \frac{(n(r-1)+2\sigma r)(1+\alpha)}{(n-2\sigma r)(1+\alpha)+2\sigma r}. \end{aligned}$$

Then there exists a positive constant ε such that for any data

$$u_0 \in L^r(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \text{ with } \|u_0\|_{L^r \cap L^\infty} \leq \varepsilon$$

we have a uniquely determined global (in time) weak solution

$$u \in C\big([0,\infty), L^r(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)\big) \text{ for all } q \in [r,\infty)$$