Ann. of Appl. Math.
35:1(2019), 99-110

STUDY OF THE STABILITY BEHAVIOUR AND THE BOUNDEDNESS OF SOLUTIONS TO A CERTAIN THIRD-ORDER DIFFERENTIAL EQUATION WITH A RETARDED ARGUMENT*

A.M. Mahmoud† D.A.M. Bakhit

(*Dept. of Math., Faculty of Science, New Valley Branch, Assiut University, El-Khargah 72511, Egypt*)

Abstract

Lyapunov direct method is employed to investigate the asymptotic behaviour and the boundedness of solutions to a certain third-order differential equation with delay and some new results are obtained. Our results improve and complement some earlier results. Two examples are given to illustrate the importance of the topic and the main results obtained.

Keywords delay differential equations; asymptotic behaviour; stability; third-order differential equations; Lyapunov functional

2000 Mathematics Subject Classification 34D20; 34K25; 34C11

1 Introduction

Differential equations (DEs) are used as tools for mathematical modeling in many fields of life science. When a model does not incorporate a dependence on its past history, it generally consists of so-called ordinary differential equation (ODEs). Models incorporating past history generally include delay differential equations (DDEs) or functional differential equations (FDEs). In applications, the future behaviour of many phenomena is assumed to be described by the solutions of an DDEs, which implies that the future behaviour is uniquely determined by the present and independent phenomena of the past. In FDEs, the past exerts its influence in a significant manner upon the future. Many phenomena are more suitable to be described by DDEs than ODEs. In many processes including physical, chemical, political, economical, biological, and control systems, time-delay is an important factor. In particular the third-order delay differential equations usually describe the phenomena in various areas of applied mathematics and physics, for instance deflection of

*Manuscript received July 28, 2018; Revised December 17, 2018

†Corresponding author. E-mail: math_ayman27@yahoo.com

bucking beam with a variable cross-section, electromagnetic waves, gravity driven flows, etc.

As we know the study of qualitative properties of solutions, such as stability and boundedness, is very important in the theory of differential equations. Since it is difficult to solve solutions to DEs, Lyapunov method is usually used to study the stability and boundedness of the equations.

Many good results have been obtained on the qualitative behaviour of solutions to some kinds of third-order DDEs by Zhu [23], Sadek [14–16], Abou-El-Ela et al. [1], Tunç [18–22], Ademola et al. [2,3], Afuwape and Omeike [4], Bai and Guo [5], Shekhare et al. [17], Remili et al. [11], and the references therein.

Numerous authors have obtained some very interesting results about the asymptotic behaviour of solutions to third-order DDEs, for example, Chen and Guan [7], Mahmoud [8], Remili et al. [10, 12, 13], etc.

In 2016, Remili and Oudjedi [12] studied the ultimate boundedness and the asymptotic behaviour of solutions to a third-order nonlinear DDE of the form

$$[\Omega(x,x')x'']'+(f(x,x')x')'+g(x(t-r(t)),x'(t-r(t)))+h(x(t-r(t)))=p(t,x,x',x''),$$

where f,g,h,Ω and p are continuous functions in their respective arguments with $g(x,0)=h(0)=0$.

In 2017, Remili et al. [13] investigated the stability and ultimate boundedness of solutions to a kind of third-order DDE as follows

$$[g(x''(t))x''(t)]'+(h(x'(t))x'(t))'+(\phi(x(t))x(t))'+f(x(t-r))=e(t),$$

where $r>0$ is a fixed delay; e,f,g,h and ϕ are continuous functions in their respective arguments with $f(0)=0$.

The main objective of this research is to study the asymptotic stability and the boundedness of solutions to a nonlinear third-order DDE

$$\begin{aligned}&[h(x(t))x''(t)]'+[p(x(t))x'(t)]'+g(x'(t-r(t)))+f(x(t-r(t)))\\&=e(t,x(t),x'(t),x''(t)),\end{aligned}\tag{1.1}$$

where h,p,g,f and e are continuous functions with $g(0)=f(0)=0$, and the derivatives $h'(u)=\frac{\mathrm{d}h}{\mathrm{d}u}$ and $p'(u)=\frac{\mathrm{d}p}{\mathrm{d}u}$ exist and are also continuous.

We can take

$$h'(x(t))x'(t)=\theta_1,\quad p'(x(t))x'(t)=\theta_2.\tag{1.2}$$

Remark In equation (1.1), if $h(x(t))=1$ and $p(x(t))=a$, then equation (1.1) is reduced to the equation in Sadek [14].