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An Integrated Quadratic Reconstruction for Finite Volume Schemes to Scalar Conservation Laws in Multiple Dimensions

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Abstract. We proposed a piecewise quadratic reconstruction method in multiple dimensions, which is in an integrated style, for finite volume schemes to scalar conservation laws. This integrated quadratic reconstruction is parameter-free and applicable on flexible grids. We show that the finite volume schemes with the new reconstruction satisfy a local maximum principle with properly setup on time steplength. Numerical examples are presented to show that the proposed scheme attains a third-order accuracy for smooth solutions in both 2D and 3D cases. It is indicated by numerical results that the local maximum principle is helpful to prevent overshoots in numerical solutions.

AMS subject classifications: 65M08, 76M12, 90C20

Key words: Quadratic reconstruction, finite volume method, local maximum principle, scalar conservation law, unstructured mesh.

1 Introduction

The study of robust, accurate, and efficient finite volume schemes for conservation laws is an active research area in computational fluid dynamics. It was noted that higher-order finite volume methods have been shown to be more efficient than second-order methods [1]. The key element in the reconstruction procedures of high-order schemes is suppressing non-physical oscillations near discontinuities, while achieving high-order accuracy in smooth regions. One of the pioneering work in this area is the finite volume scheme based on the k-exact reconstruction, first proposed by Barth and Fredrichson [2]

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and later extended to the cell-centered finite volume scheme by Mitchell and Walters [3]. For more recent work on the use of k-exact reconstruction to attain high-order accuracy, we refer the reader to [1,4–7] for instance. The hierarchical reconstruction strategies of Liu *et al.* [8] were also used to achieve higher-order accuracy [9, 10], where the information is recomputed level by level from the highest order terms to the lowest order terms with certain non-oscillatory method. Other type of high-order finite volume schemes includes the WENO scheme [11]. Although the implementation of WENO scheme is comparatively complicated on unstructured meshes due to the needs of identifying several candidate stencils and performing a reconstruction on each stencil [6], it has been successfully applied on the unstructured meshes for both two-dimensional triangulations [12–17] and three-dimensional triangulations [18, 19]. Most of these schemes do not lead to a strict maximum principle, while they are essentially non-oscillatory [20]. Actually, the reconstruction procedure for maximum-principle-satisfying second-order schemes are relatively mature [21–26], while there are few maximum-principle-satisfying reconstruction approaches for higher-order finite volume schemes on unstructured meshes.

Limiting to scalar conservation laws, a quadratic reconstruction for finite volume schemes applicable on 2D and 3D unstructured meshes is developed in this paper. The construction is a further exploration of the integrated linear reconstruction (ILR) in [27, 28], where the coefficients of reconstructed polynomial are embedded in an optimization problem. It is appealing for us to generalize the optimization-based constructions therein to an integrated quadratic reconstruction (IQR) such that the scheme achieves a third-order accuracy while satisfying a local maximum principle. It was pointed out in [29] that the scheme satisfying the standard local maximum principle is at most second-order accurate around extrema. To achieve higher than second-order accuracy, high-order information of the exact solution has to be taken into account in the definition of local maximum principle [29, 30]. Sanders [31] suggested to measure the total variation of approximation polynomials. Liu *et al.* [32] constructed a third-order non-oscillatory scheme by controlling the number of extrema and the range of the reconstructed polynomials. Zhang *et al.* constructed a genuinely high-order maximum-principle-satisfying finite volume schemes for multi-dimensional nonlinear scalar conservation laws on both rectangular meshes [33] and triangular meshes [34] by limiting the reconstructed polynomials around cell averages. The flux limiting technique developed by Christlieb *et al.* [35] is another family of maximum-principle-satisfying methods on unstructured meshes. In our scheme, it is proposed that the extrema of numerical solutions are measured by extrema of polynomial on a cluster of points, following the technique of Zhang *et al.* [33, 34]. To overcome the difficulty of loss of high-order information in cell averages, besides the cell averages at current time level, we utilize the reconstruction polynomials at previous time step. This idea is based on the wave propagation nature of conservation laws. Since the solution value at $(\mathbf{x},t)$ can be tracked back to a point in the ball around $\mathbf{x}$ with radius as $v\Delta t$ at time $t-\Delta t$, while v is local wave speed, it is reasonable for us to use the value in this ball at previous time step in the reconstruction. It is shown that this may lead to a third-order numerical scheme, meanwhile a local maximum principle is satisfied. An