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A Diagonal Sweeping Domain Decomposition Method with Source Transfer for the Helmholtz Equation

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Abstract. In this paper, we propose and test a novel diagonal sweeping domain decomposition method (DDM) with source transfer for solving the high-frequency Helmholtz equation in $\mathbb{R}^n$. In the method the computational domain is partitioned into overlapping checkerboard subdomains for source transfer with the perfectly matched layer (PML) technique, then a set of diagonal sweeps over the subdomains are specially designed to solve the system efficiently. The method improves the additive overlapping DDM [43] and the L-sweeps method [50] by employing a more efficient subdomain solving order. We show that the method achieves the exact solution of the global PML problem with 2^n sweeps in the constant medium case. Although the sweeping usually implies sequential subdomain solves, the number of sequential steps required for each sweep in the method is only proportional to the n-th root of the number of subdomains when the domain decomposition is quasi-uniform with respect to all directions, thus it is very suitable for parallel computing of the Helmholtz problem with multiple right-hand sides through the pipeline processing. Extensive numerical experiments in two and three dimensions are presented to demonstrate the effectiveness and efficiency of the proposed method.

AMS subject classifications: 65N55, 65F08, 65Y05

Key words: Helmholtz equation, domain decomposition method, diagonal sweeping, perfectly matched layer, source transfer, parallel computing.

1 Introduction

In this paper, we consider the well-known Helmholtz equation defined in $\mathbb{R}^n$ ($n=2,3$) as follows:

$$\Delta u+\kappa^2 u=f, \qquad \text{in } \mathbb{R}^n, \tag{1.1}$$

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imposed with the Sommerfeld radiation condition

$$r^{\frac{n-1}{2}}\left(\frac{\partial u}{\partial r}-\mathbf{i}\kappa u\right)\rightarrow 0, \qquad \text{as } r=|\mathbf{x}|\rightarrow\infty, \tag{1.2}$$

where $u(\mathbf{x})$ is the unknown function, $f(\mathbf{x})$ is the source and $\kappa(\mathbf{x}):=\omega/c(\mathbf{x})$ denotes the wave number with ω being the angular frequency and $c(\mathbf{x})$ the wave speed. Solving the Helmholtz equation (1.1) with large wave number accurately and efficiently is crucial to many physics and engineering problems. For example, in exploration seismology, the Helmholtz equation with pre-given wave speed needs to be solved for hundreds of different sources in reverse time migration, and even more in full wave inversion. However, since the discrete Helmholtz system with large wave number is highly indefinite, constructing efficient solvers is quite important and challenging [27], and for this purpose many methods have been proposed and studied, including the direct method [19], the multigrid method [26] and the domain decomposition method [16].

The direct method, such as the multifrontal method [19] with nested dissection [37], was designed to solve linear systems arising from discretization of general PDE problems, and has been employed to solve the discrete Helmholtz problem. The multifrontal method was further coupled with the hierarchically semi-separable matrices (HSS) in [39], and the low rank proprieties were exploited to reduce the computational complexity for many problems including Helmholtz equation in [55, 56]. However, the low-rank representation for the Helmholtz kernel in high frequency is missing [22], which causes the HSS and multifrontal coupled method to be less effective for high frequency problems. On the other hand, some variants of the multifrontal method were also introduced in [38, 44] for the Helmholtz problem. Those methods mostly focus on constructing the Dirichlet to Neumann (DtN) map for the subdomains in the nested dissection, which is more intuitive than manipulating the algebraic matrices in the multifrontal method, while the order of computational complexity remains the same.

The multigrid method with the shifted Laplace was first introduced in [26], and then further developed in [3, 23–25, 48, 53]. A complex shift is added to the Helmholtz operator, resulting in an easier problem that could be solved with multigrid solver, which then can be used as an effective preconditioner for the original Helmholtz problem. The shifted Laplace method has been shown to be very effective, and followed by many researches in literature, to name a few, [1, 7, 9, 14, 15, 34, 40, 46, 52]. The amount of the shift is a compromise, a larger shift leads to an easier problem to solve in preconditioning but more iteration steps in the Krylov subspace solve, while a smaller shift results in harder preconditioning but fewer iteration steps. For the high frequency problem, if the shifted problem in preconditioning is required to be solved efficiently, then the number of iterations in the Krylov subspace solve grows as fast as the square of the frequency [35], thus the high frequency problem still remains a big challenge for the shifted Laplace method. The two-level domain decomposition preconditioner [8] is effective for some complicated applications such as elastic crack problems, but applying such method to the Helmholtz problem is quite difficult since it is very hard to construct a proper coarse space that