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A Novel Numerical Approach to Time-Fractional Parabolic Equations with Nonsmooth Solutions

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Abstract. This paper is concerned with numerical solutions of time-fractional parabolic equations. Due to the Caputo time derivative being involved, the solutions of equations are usually singular near the initial time $t = 0$ even for a smooth setting. Based on a simple change of variable $s = t^\beta$, an equivalent s-fractional differential equation is derived and analyzed. Two type finite difference methods based on linear and quadratic approximations in the s-direction are presented, respectively, for solving the s-fractional differential equation. We show that the method based on the linear approximation provides the optimal accuracy $\mathcal{O}(N^{-(2-\alpha)})$ where N is the number of grid points in temporal direction. Numerical examples for both linear and nonlinear fractional equations are presented in comparison with $L1$ methods on uniform meshes and graded meshes, respectively. Our numerical results show clearly the accuracy and efficiency of the proposed methods.

AMS subject classifications: 35A35, 35R11, 65M12
Key words: Time-fractional differential equations, nonsmooth solution, finite difference methods, $L1$ approximation.

1. Introduction

Time-fractional differential equations have attracted much attention in the last two decades since many physical models can be described more precisely in this way. Here,

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we consider the time-fractional parabolic equations in the form

$$\mathcal{D}_t^\alpha u + \mathcal{L}u = f, \quad x \in \Omega \times (0,T] \tag{1.1}$$

with the initial and boundary conditions given by

$$\begin{aligned} &u(x,0) = u_0(x), && x \in \Omega, \\ &u(x,t) = 0, && x \in \partial\Omega \times [0,T], \end{aligned} \tag{1.2}$$

where $\mathcal{L}$ is a second-order linear and strongly elliptic differential operator on $\bar{\Omega}$. Since we mainly focus on finite difference discretization, we simply assume that $\Omega = [0,b]^d$, where d denotes the dimension. The Caputo fractional derivative $\mathcal{D}_t^\alpha$ is defined by

$$\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,z)}{\partial z} \frac{1}{(t-z)^\alpha} dz, \quad 0 < \alpha < 1, \tag{1.3}$$

where $\Gamma(\cdot)$ denotes the usual Gamma function.

Numerous effort has been devoted to developing effective numerical methods and rigorous numerical analysis for the time fractional differential equation (1.1)-(1.2). Clearly, the accuracy of numerical methods heavily relies on the regularity of the solution of the equation. Theoretical analysis based on assumption of the solution being smooth was done by many authors for different applications and different numerical methods. Unlike regular differential equations ($\alpha = 1$), the solution of the time-fractional differential equations may not be smooth even for a smooth setting (for source term, the boundary/initial conditions and compatibility conditions), see [5, 6, 15,16,24,28,30,36,37] for detailed discussion. For certain simple time-independent elliptic operator $\mathcal{L}$, with a standard separation of variable, the solution of (1.1)-(1.2) can be given [35] in terms of the expansion of eigenpairs $(\lambda_k, \psi_k(x))$ of the corresponding steady state problem by

$$u(x,t) = \sum_{k=1}^{\infty} \big[(u_0, \psi_k) E_{\alpha,1}(-\lambda_k t^\alpha) + J_k(t)\big] \psi_k(x), \tag{1.4}$$

where

$$\begin{aligned} J_k(t) &= \int_0^t z^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k z^\alpha) f_k(t-z) dz, \\ f_k(t) &= \int_\Omega f(x,t)\psi_k(x) dx, \quad E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \end{aligned}$$

defines the classical Mittag-Leffler function. More details can be found in [36]. From (1.4) one can see that the solution has a singular layer near $t = 0$, in which the optimal error estimate of the $L1$ scheme on a uniform temporal mesh is [14, 37]

$$\max_{1 \le n \le N} \|e^n\| \le \mathcal{O}(\tau^\alpha). \tag{1.5}$$