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Convergence of the Peaceman-Rachford Splitting Method for a Class of Nonconvex Programs

Miantao Chao[1,2], Deren Han[3] and Xingju Cai[1,*]

[1] *School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China*
[2] *College of Mathematics and Information Science, Guangxi University, Nanning 530004, China*
[3] *School of Mathematics and System Sciences, Beihang University, Beijing 100191, China*

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Abstract. In this paper, we analyze the convergence of the Peaceman-Rachford splitting method (PRSM) for a type of nonconvex and nonsmooth optimization with linear constraints, whose objective function is the sum of a proper lower semicontinuous function and a strongly convex differential function. When a suitable penalty parameter is chosen and the iterative point sequence is bounded, we show the global convergence of the PRSM. Furthermore, under the assumption that the associated function satisfies the Kurdyka-Łojasiewicz property, we prove the strong convergence of the PRSM. We also provide sufficient conditions guaranteeing the boundedness of the generated sequence. Finally, we present some preliminary numerical results to show the effectiveness of the PRSM and also give a comparison with the Douglas-Rachford splitting method.

AMS subject classifications: 90C26, 90C30

Key words: Kurdyka-Łojasiewicz inequality, Peaceman-Rachford splitting method, nonconvex, strongly convex, Douglas-Rachford splitting method.

1. Introduction

Consider the following optimization problem

$$\begin{aligned} \min \quad & f(x) + g(y) \\ \text{s.t.} \quad & Ax + By = b, \end{aligned} \tag{1.1}$$

where the function $f\colon \mathcal{R}^n \to \mathcal{R} \cup \{+\infty\}$ is proper and lower semicontinuous, the

*Corresponding author. *Email address:* `caixingju@njnu.edu.cn` (X. Cai)

function $g\colon \mathcal{R}^m \to \mathcal{R}$ is strongly convex and differentiable, ∇g is Lipschitz continuous with modulus $L > 0, A \in \mathcal{R}^{l\times n}, B \in \mathcal{R}^{l\times m}$ and $b \in \mathcal{R}^l$. Many applications can be modeled in the form of problem (1.1), e.g., sparse logistic regression problems [19] and $\ell_{(1/2)}$-norm regularized least-squares problems [28].

A classic algorithm for solving problem (1.1) is the Peaceman-Rachford splitting method (PRSM) [18, 22]. The PRSM was originally introduced in [22] for solving linear heat flow equations and was later generalized to address nonlinear equations in [19]. Another effective splitting method for solving problem (1.1) is the Douglas-Rachford splitting method (DRSM) [3, 4, 18]. Recently, the DRSM was adapted to the nonconvex setting [10, 11, 14, 16, 24–26, 29] under suitable assumptions. Although, the PRSM can be faster than the DRSM when it is convergent (see, [6]), the PRSM is not as popular as the DRSM in the convex case. One of the main reasons is that, in the convex setting, the convergence condition of PRSM is stronger than that of DRSM. Thus, it is interesting to study the convergence of the PRSM in the nonconvex case. In [15], Li *et al.* employed the classic PRSM to solve the following nonconvex problem

$$\min\ f(x) + g(x),$$

where the function f is differentiable and strongly convex with modulus $\sigma > 0, \nabla f$ is Lipschitz continuous with modulus $L > 0$. They showed that if $3\sigma > 2L$ and the stepsize parameter is below a computable threshold, then any cluster point of the generated sequence, if it exists, will give a stationary point of the optimization problem. The procedure of the classic PRSM for (1.1) is as follows:

$$\begin{cases} x^{k+1} = \underset{x}{\operatorname{argmin}}\, \mathcal{L}_\beta\big(x, y^k, \lambda^k\big), & \text{(1.2a)} \\ \lambda^{k+\frac{1}{2}} = \lambda^k - \beta\big(Ax^{k+1} + By^k - b\big), & \text{(1.2b)} \\ y^{k+1} \in \underset{y}{\operatorname{argmin}}\, \mathcal{L}_\beta\big(x^{k+1}, y, \lambda^{k+\frac{1}{2}}\big), & \text{(1.2c)} \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta\big(Ax^{k+1} + By^{k+1} - b\big), & \text{(1.2d)} \end{cases}$$

where

$$\mathcal{L}_\beta(x, y, \lambda) = f(x) + g(y) - \lambda^T(Ax + By - b) + \frac{\beta}{2}\|Ax + By - b\|^2 \tag{1.3}$$

is the augmented Lagrangian function of (1.1), $\lambda \in \mathcal{R}^l$ is the Lagrange multiplier and $\beta > 0$ is a penalty parameter.

Wu *et al.* [27] proposed the following modified PRSM for solving (1.1) with B as an identity matrix, whose iterative scheme is

$$\begin{cases} x^{k+1} \in \underset{x}{\operatorname{argmin}}\, \mathcal{L}_\beta\big(x, y^k, \lambda^k\big), & \text{(1.4a)} \\ \lambda^{k+\frac{1}{2}} = \lambda^k - \alpha\beta\big(Ax^{k+1} + By^k - b\big), & \text{(1.4b)} \\ y^{k+1} = \underset{y}{\operatorname{argmin}}\, \mathcal{L}_\beta\big(x^{k+1}, y, \lambda^{k+\frac{1}{2}}\big), & \text{(1.4c)} \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta\big(Ax^{k+1} + By^{k+1} - b\big). & \text{(1.4d)} \end{cases}$$